Probability Models.S5
Exercises

1. From the daily newspaper identify five quantities that are variable in time and uncertain for the future. Provide data or charts that describe the historical variability. Describe decisions that are affected by the values of the quantities.

2. A group of six students draw tickets to the football game. The tickets drawn are for six adjacent seats all on the same row. The tickets are passed out at random.
   One boy and one girl have a secret affection for each other. If the tickets are passed out at random, what is the probability that the boy and girl will sit next to each other?

3. A repeatable experiment involves observing two numbers a and b. The possible observations are represented by number pairs (a, b) and include:

   $\boldsymbol{S}$ = {(1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2)}.

   The twelve observations are equally likely. Given the events:

   $\boldsymbol{W} = \{(a, b) | a = 1\}$, $\boldsymbol{X} = \{(a, b) | b < 2\}$, $\boldsymbol{Y} = \{(a, b) | a + b \le 4\}$, $\boldsymbol{Z} = \{(a, b) | b = 2\}$

   Find the probabilities of the events: ***W***, ***X***, ***Y***, ***Z***, ***W***   ***Y***, $\boldsymbol{W} \cap \boldsymbol{Z}$, ***W***   ***X***   ***Y***   ***Z***.

4. This problem uses the events ***W***, ***X***, ***Y***, and ***Z*** defined in Problem 3. Again all outcomes are equally likely. Consider in turn each of the six pairs that can be formed from these events. For each pair, indicate whether the pair is mutually exclusive or independent.

   Pairs: ***W*** and ***X***, ***W*** and ***Y***, ***W*** and ***Z***, ***X*** and ***Y***, ***X*** and ***Z***, ***Y*** and ***Z***.

5. Five events are defined on a sample space: ***A***, **B**, **C**, **D**, and **E**. Events **A**, **B**, and **C** are mutually exclusive. The following probabilities are given:

   P(**A**) = 0.1, P(**B**) = 0.4, P(**C**) = 0.5, P(**D**) = 0.16, P(**D**|**A**) = 0.8,
   P(**D**|**B**) = 0.2, P(**D**|**C**) = 0, P(**E**|**A**) = 0, P(**E**|**B**) = 0.9, P(**E**|**C**) = 0.1.

   Compute the following probabilities:

   P(**D** ∩ **A**), P(**B**|**D**), P(**D**   **B**), P(**E**), P(**B**|**E**).

6. A quality control test involves removing a sample of three products from the production line in each hour. Each of the three units is tested to determine if it meet specifications or not. The random variable is the number of units that meet specifications.
   Use the events: ***A***: At most 2 meet the specifications, ***B***: Exactly 3 meet the specifications, and ***C***: At least 2 meet the specifications.
   a. Which events are mutually exclusive and which are not?
   b. In terms of the random variable, what are the events ***A***   ***B***, ***A***   ***C***, and $\boldsymbol{A} \cap \boldsymbol{C}$.
   c. Say each unit meets specifications with probability 0.95 and that the units are independent. Assign a probability to each event (and the combinations of events) considered in parts a and b.

7. The random variable X is the number of cars entering the campus from 1 to 1:05 A.M. Assign probabilities according to the formula:

$$P(X = k) = \frac{e^{-5}(5^k)}{k!} \quad \text{for } k = 0,\ 1,\ 2,\ \ldots$$

a. Using the events: ***A***: More than 5 cars enter, ***B***: Fewer than 3 cars enter, and ***C***: Between 4 and 8 cars enter. Find:
$P(\boldsymbol{A})$, $P(\boldsymbol{B})$, $P(\boldsymbol{C})$, $P(\boldsymbol{A}\ \ \boldsymbol{B})$, $P(\boldsymbol{A}\ \ \boldsymbol{C})$, $P(\boldsymbol{B}\ \ \boldsymbol{C})$, $P(\boldsymbol{A}\ \ \boldsymbol{B}\ \ \boldsymbol{C})$, $P(\boldsymbol{A}\cap\boldsymbol{B})$, $P(\boldsymbol{A}\cap\boldsymbol{C})$, $P(\boldsymbol{B}\cap\boldsymbol{C})$, $P(\boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C})$, $P(\boldsymbol{A}|\boldsymbol{B})$, $P(\boldsymbol{C}|\boldsymbol{A})$, $P(\boldsymbol{A}|\boldsymbol{C})$.

b. Compute the c.d.f. associated with the random variable.

8. Assign probabilities to an event $\boldsymbol{E} = \{t \mid t_1 < t < t_2\}$ with the function

$$P(\boldsymbol{E}) = (exp(-t_1) - exp(-t_2))$$

Define $\boldsymbol{A} = \{t \mid 0 < t < 1\}$, $\boldsymbol{B} = \{t \mid 1.5 < t < \infty\}$ and $\boldsymbol{B} = \{t \mid 0.5 < t < 1.5\}$

Find: $P(\boldsymbol{A})$, $P(\boldsymbol{B})$, $P(\boldsymbol{C})$, $P(\boldsymbol{A}\ \ \boldsymbol{B})$, $P(\boldsymbol{A}\ \ \boldsymbol{C})$, $P(\boldsymbol{B}\ \ \boldsymbol{C})$, $P(\boldsymbol{A}\ \ \boldsymbol{B}\ \ \boldsymbol{C})$, $P(\boldsymbol{A}\cap\boldsymbol{B})$, $P(\boldsymbol{A}\cap\boldsymbol{C})$, $P(\boldsymbol{B}\cap\boldsymbol{C})$, $P(\boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C})$, $P(\boldsymbol{A}|\boldsymbol{B})$, $P(\boldsymbol{C}|\boldsymbol{A})$, $P(\boldsymbol{A}|\boldsymbol{C})$.

9. Consider the situation of throwing two dice (each die has sides numbered 1 through 6). For each of the following random variables find: $P(3 \leq X \leq 5)$.
   a. X = The sum of the two dice.
   b. X = The number on the first die.
   c. X = The average of the two dice.
   d. X = The largest of the two numbers on the dice.

10. A teenager makes money by babysitting on Saturday nights. For each job she makes $15. Based on past experience, she computes that there is an 80% chance that she will be called for a job in one week. The weeks are independent. The girl wants to save $90 for a new dress. Assuming she saves all the money she makes, what is the probability that more than eight weeks are required to save the $90?

11. Computations indicate that a missile fired at a target will destroy the target with a 0.35 probability. Assume all missiles fired are independent and have the same probability of destruction. How many missiles must be fired to assure that the target will be destroyed with a 0.90 probability?

12. The kicker on the football squad, has a 60% chance of successfully kicking a field goal on each try. If he is given five opportunities in a game, what is the probability that he will complete at least four field goals in the game?

13. Ten students in a class of 30 are girls. Four of the top five grades in an exam are scored by girls. What is the probability that this result could have occured purely by chance?

14. During the preregistration period, students arrive at random to a professor's office at an average rate of 5 per hour. What is the probability that fewer than two students will arrive during a ten minute period?

15. We select an item from the assembly line and note that it is either operating or failed. The associated probabilities are: *P*(Operating) = 0.8 and *P*(Failed) = 0.2. For each situation identify the probability distribution, specify the parameters, and answer the question.
    a. We draw 10 such items from the line, what is the probability that a majority of them will work?
    b. What is the probability that at least 10 items must be taken from the line before we find five working items?
    c. What is the probability that we will draw 5 working items before the first failed one is encountered?

16. A standard deck of poker cards has 52 cards. In the deck there are 13 different card types labeled: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. Each type has 4 cards, each in a different suit: clubs, diamonds, hearts and spades.
    a. I am about to be dealt a hand of five cards (this is selection without replacement). What is the probability that at least three of the cards will be Aces?
    b. The dealer will show me five cards. After each card is drawn, it is returned to the deck (this is selection with replacement). What is the probability that at least three of the cards shown are Aces?
    c. The dealer will show me a series of cards. After each card is drawn, it is returned to the deck (this is selection with replacement). The dealer will keep showing me the cards until 3 aces are discovered. What is the probability that five cards or fewer cards must be shown?

17. A professor always gives tests made up of 20 short answer questions. All the questions give 5 points credit for a total score of 100 if all are answered correctly. She grades the questions as either right or wrong with no partial credit. Based on your current capability you judge that you have a 0.7 probability of answering any one question correctly. Assume the questions are independent in terms of chance of success. If a score of 80 or more earns at least a B, what is the probability that you will earn at least a B for this exam?
18. A professor always gives tests made up of 20 short answer questions. All the questions give 5 points credit for a total score of 100 if all are answered correctly. She does give partial credit. Based on your current capability you judge that your credit on each question is a random variable with mean equal to 3 and a standard deviation equal to 1. Assume the questions are independent in terms of chance of success. Before you take the test what is your estimate of the mean and standard deviation of your test score? If a score of 80 or more earns at least a B, what is the probability that you will earn a B for this exam? You will have to make an approximation to answer this question.

19. A major company needs three engineers but will accept more. The manager in charge of hiring will conduct a series of interviews in hopes of finding at least three “good” applicants who will accept the offer of a job. The probability that an applicant is “good” is

0.4, and the probability that a "good" applicant will accept the job is 0.6. She makes offers to every "good" applicant. We assume applicants are independent.

a. If the manager conducts 10 interviews, what distribution describes the number of engineers she will hire? What is the probability that at least three will be hired?

b. If the manager conducts as many interviews as necessary to find the three engineers, what distribution describes the number of interviews required? What is the probability that at least 10 interviews are required?

20. A robot insertion tool must place 4 rivets on a printed circuit board. Rivets are presented to it at random, but 10% of the rivets are faulty. When a faulty rivet is encountered it must be discarded, and another drawn. We want to compute probabilities regarding the number of draws required before 4 good rivets are found. In particular what is the probability that exactly 10 draws are required before we find 4 good rivets.

21. The faculty of a department at the university has six full professors, four associate professors, and five assistant professors. A committee with five members is chosen at random from the professors. What is the probability that a majority of the committee will be full professors.

22. A box contains three red dice, two blue dice, and one green die. For the following experiments describe the probability distribution function for the random variable.

a. Three dice are drawn from the box with replacement. The random variable is the number of red dice drawn. What is the probability that at least 2 red dice are drawn?

b. Three dice are drawn without replacement. The random variable is the number of blue dice drawn. What is the probability that at exactly 2 blue dice are drawn?

c. Draw a die out of the box. Observe its color. Return the die to the box. Continue drawing dice in this manner until the green die is found or five dice have been drawn. The random variable is the number of red or blue dice drawn before the experiment stops. What is the probability that the process stops in three or less draws.

23. The game of Russian Roulette uses a gun which has a single live bullet placed in one of the six places in the cylinder. Player A spins the cylinder, places the gun to his head and pulls the trigger. If player A survives, player B repeats the process. The game continues with the players taking turns until one finds the bullet.

a. Compute the probability distribution of the number of times the trigger will be pulled until the player is killed. The last pull kills one of the players.

b. Find the probability that the game lasts longer than six plays.

c. If the players pledge to stop after six plays, what is the chance that each player will be killed.

24. A line forms in front of a ticket booth with one window. We have observed the number of people in the line (including the one being served). Our data estimates the probabilities for i persons in the line, p_i , where $i = 0, 1, \ldots 7$. We never observe more than four persons in the line. The probability distribution is shown below.

k	0	1	2	3	4
p_k	0.3278	0.2458	0.1844	0.1383	0.1037

a. Compute the mean value and standard deviation of the number of persons in the line.
b. Compute the average number of servers that are busy.
c. A measure of the effectiveness of a queueing system is the utilization of the servers, that is the proportion of the time the servers are busy. What is the utilization of the servers in this system?

25. Find the values of the mean and standard deviations for the following distributions:
a. $f(x) = 1/4$ for $x = 1, 2, 3, 4$.
b. x has a binomial distribution with $n = 10$ and $p = 0.6$.
c. $f(x) = \dfrac{e^{-4}(4)^x}{x!}$ for $x = 0, 1, 2, \ldots$

26. A continuous random variable has the p.d.f.

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}.$$

Derive the formulas for the c.d.f., mean, variance, and median.

27. Consider a continuous random variable with the p.d.f.

$$f(x) = \begin{cases} 0.5 & \text{for } 0 \le x \le 0.5 \\ k & \text{for } 0.5 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

What is the value of k that makes this a legitimate p.d.f? Find an expression for the c.d.f. for this random variable.

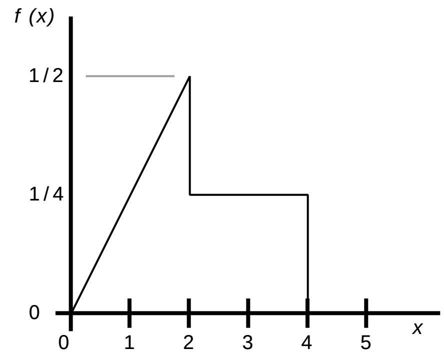
28. Consider a continuous random variable with the p.d.f.

$$f(y) = \begin{cases} k(y - 10) & \text{for } 10 \le x \le 20 \\ 0 & \text{elsewhere} \end{cases}.$$

a. Find the value of k that makes this a p.d.f. Find an expression that describes $F(y)$ for all y.
b. Compute the mean, standard deviation, median, mode of the distribution.
c. From the c.d.f. find: $P\{y < 12\}$, $P\{12 < y < 18\}$.

29. We want to do problem 28 by expressing Y as a linear transformation of a random variable X. X has a triangular distribution which ranges from 0 to 1.
a. What is c for the distribution of X?
b. What are a and b for the transformation?
c. Compute the mean, standard deviation, median, mode of Y from the corresponding values for X.
d. From the c.d.f. for X find: $P\{Y < 12\}$, $P\{12 < Y < 18\}$.

30. For the probability distribution, find
a. $P(1 \le x \le 3)$, b. The mean, c. The variance, d. The median, e. The mode.



31. A manufacturer of bathrobes for men makes only one size of robe. The robe is designed to fit any man with a chest size more than 39 inches and less than 43 inches. Using data gathered by the government, the company determines that the mean chest size of men in the United States is 41.5 inches. The standard deviation is 1.75 inches. Assuming that chest size has a normal distribution, what proportion of the population will not fit into the robe?

32. Consider again the information given in problem 31. In addition to the information given there, it is determined that the robe will not fit men who are less than 68 inches tall or more than 75 inches tall. Assume that the height of men has a normal distribution with mean 70 inches and a standard deviation of 3 inches. For simplicity assume also that height and chest size are independent random variables. What proportion of the men will fit into the robe considering both kinds of restrictions?

33. A grocery store clerk must pack n items in a bag. The average time to pack an item is 1 second with a standard deviation of 0.5 seconds. What is an appropriate distribution for modeling the total time required to pack the bag.

34. The last piece is never quite right. Sister is cutting a pie into six pieces. Starting from a given point in the pie she can will cut a slice of pie with a central angle α. The central angle is a random variable with Normal distribution with mean equal to 60 degrees and standard deviation equal to 5 degrees. The first five pieces are cut with this accuracy. The last piece is the remainder after the first five pieces are cut. If the family judges as acceptable a piece with an angle between 55 and 65 degrees. What is the probability that each of the first five pieces is acceptable? What is the probability that the final piece is acceptable? What is the probability that all six pieces are acceptable?

35. Consider a restaurant open only for the lunch hour. The number of persons seeking service at the restaurant, Z, has a Normal distribution with mean μ_Z and standard deviation σ_Z. Because of congestion in the restaurant it has been found that the time for the restaurant to clear after the lunch hour, X, is an exponential function of the number of customers. That is

$$X = a[exp(bZ)], \text{ where } a \text{ and } b \text{ are constants.}$$

What is the probability distribution for the random variable X? In terms of μ_Z , σ_Z, a and b, what are the mean and standard deviation of X?

36. A manufacturing operation has a mean time of 10 minutes. We have the option of dividing the operation into 1, 2, 5 or 10 components. The components must be accomplished

sequentially and the entire operation is complete when all the components are finished. When the number of components is n, the mean time to complete a component is $10/n$. The time to complete each component has an exponential distribution. Compute the probability that the total completion time is between 9 and 11 minutes.

37. Use the Standard Normal distribution to approximate the case of ten components in problem 36. Compute the $P(9 < Y < 11)$ and compare it with the value computed with the Gamma distribution.

38. An expert estimates that a project will take at least four months and at most eight months. We would like to associate a generalized Beta distribution with these estimates. For the several possible values of the most likely times given in the table, fill in the missing values of α and β.

Most likely	4	5	6	7	8
α		2	2		2
β	2			2	

39. An expert estimates that a project will take at least four months and at most eight months. Find the probability that the completion time will be less than six months when we use the following Beta distribution parameters to model the time to completion.

a. $\alpha = 2$, $\beta = 2$; b. $\alpha = 2$, $\beta = 5$; c. $\alpha = 5$, $\beta = 2$.

40. Ten students join hands. The reach of one student (the distance from the left to the right hand) is a random variable with a normal distribution having $\mu = 38$ inches and $\sigma = 4$ inches. What is the probability distribution of the reach of all ten students together?

41. A student enters a bank and finds a single teller at work and a line that forms in front of the teller. From past experience the student estimates a mean time of two minutes for each customer. What is the probability that she will begin her service within the next 15 minutes if there is one person ahead of her (the person is already in service)? Answer this question if she finds two persons waiting or in service when she enters the bank. Answer the question for 5 persons. Answer this question for the following service time distributions.

a. The time for service of each customer has an exponential distribution.
b. The time for service of each customer has a Normal distribution with a mean of 2 minutes and a standard deviation of 1 minute. Accept the possibility of negative times.
c. The time for service has a Lognormal distribution with a mean of 2 minutes and a standard deviation of 1 minute.

	Mean	2		Var	1
Solving for alpha and Betε	Beta =	0.47238073		Alpha	0.5815754

Random Var.	Wait_LN
Distribution	Lognormal
alpha	0.5815754
beta	0.47238073

Mean	2
Var	1

P(Wait > 15)	3.3751E-06	3.6637E-15	0.01267362

For two customers waiting we must simulate the sum of two Log Normal random variables.

Simulated Time	4.34270233	0	100

In 100 trials no times of over 15 minutes were observed

Replication	1.
Simulated Cell	C45
Sample Size	100.
Mean	0.

For five customers waiting we must simulate the sum of five Log Normal random variables.

Simulated Time	11.8619842	0	100

In 100 trials 2 times of over 15 minutes were observed

Replication	1.
Simulated Cell	C52
Sample Size	100.
Mean	0.02

42. A submarine must stay below the sea for a period of three months. A particularly important subsystem of the submarine has a mean time to failure of 5 months. The time to failure has an exponential distribution.

a. What is the reliability of the subsystem? The reliability is defined as the probability that the subsystem will operate successfully for the full three month mission.

b. The submarine stores one redundant subsystem (2 in all). The redundant subsystem has no failure rate while not operating. What is the reliability of the subsystem with redundancy.

c. The submarine stores four redundant subsystem (5 in all). The redundant subsystems have no failure rate while not operating. What is the reliability of the subsystem with redundancy.

d. Comment on the effect of redundancy on the reliability of the system. What are the critical assumptions that make the above analysis valid?

e. Say the cost of buying and holding a component for a mission is $1000. The cost of provisioning the submarine with new components if all on board fail is $200,000. What is the optimum number of components to bring?

f. Change the situation of this problem so that all components have the same failure rate whether they are operating or not. The sub still requires one nonfailed component for a successful mission. What is the optimum number of components to bring?

g. Change the situation of this problem so that the time to failure of the operating component has a Weibull distribution with mean equal to 5 months and $\beta = 2$. Non-operating components cannot fail. Estimate with simulation, the reliability of the system with 1, 2, and 3 components. You will have to simulate to answer this problem for 2 and 3 components. Use 500 observations.

Problems 43 - 45. The numbers in the table below were generated by a spreadsheet random number generator. Use them for the following problems.

Random Numbers

	1	2	3	4	5	6
1	0.3410	0.4921	0.5907	0.9658	0.8632	0.6327
2	0.8228	0.4294	0.0809	0.3400	0.4116	0.8931
3	0.6066	0.3841	0.8915	0.6096	0.7013	0.4947
4	0.3729	0.7230	0.1621	0.6537	0.0011	0.3888
5	0.1241	0.9864	0.7059	0.3734	0.9895	0.0768

43. Use row 1 to generate 6 observations from:

a. A Bernoulli distribution $p = 0.4$. Use the range $(0 \le r \le 0.6)$ for 0 and $(0.6 < r \le 1)$ for 1.

b. A Binomial distribution with $n = 5$ and $p = 0.4$.

c. A Geometric distribution with $p = 0.4$.

d. A Poisson distribution with $\theta = 2$.

44. Use row 1 to generate 6 observations from:

a. A Standard Normal distribution.

b. A Normal distribution with $\mu = 100$ and $\sigma = 20$.

c. A Lognormal distribution whose underlying distribution is a Standard Normal.

d. A Weibull distribution with $\theta = 2$.

45. Use all six numbers from each row to simulate five observations from the distributions below.

a. Simulate five observations of a Binomial random variable with $n = 6$ and $p = 0.4$ by simulating Bernoulli trials with $p = 0.4$.

b. Simulate five replications of the sum of six observations by summing six simulated observations from the Standard Normal distribution.

c. Simulate five observations from the Gamma distribution with $n = 6$ and $\lambda = 2$ by summing simulated exponential random variables.

46. When evaluating investments with cash flows at different times we compute the net present worth (NPW) of the cash flows. Consider an investment of the amount P with annual returns I_k for $k = 1$ to n. n is the life of the investment and I_k is the return in year k. The NPW is defined in terms of i is the investor's minimum acceptable rate of return (MARR). If NPW ≥ 0, the investment provides a return at least as great as the MARR. If NPW < 0, the investment fails to provide the MARR.

Your investment advisor tells you a particular investment of $1000 will provide a return of $500 per year for three years. Your MARR is 20% or 0.2.

a. Is your investment acceptable according to the NPW criterion.

b. The advisor adds the information that the life of the investment is uncertain. Rather there is a uniform distribution on the life with $p_n = 0.2$ for a life of $n = 1, 2, 3, 4,$ and 5. Use the random numbers below to simulate ten replications of the life and compute the NPW in each case.

0.5758	0.4998	0.3290	0.3854	0.6784	0.8579
0.5095	0.6824	0.3762	0.1351	0.2555	0.9773

Based on your simulation, what is the probability that the investment will yield the MARR?

c. The advisor now tells you that the annual revenue per year is uncertain. Although it will be the same each year, the revenue is Normally distributed with a mean equal to 500 and a standard deviation of 200. Use the random numbers below to simulate ten observations of the annual revenue.

0.5153	0.3297	0.6807	0.0935	0.9872	0.6339
0.0858	0.3229	0.5285	0.4451	0.3177	0.1562

Combine the lives simulated with the revenues of this section to determine the probability that the investment will yield the MARR.