Probability Models.S1
Introduction to Probability

The stochastic chapters of this book involve random variability. Decisions are to be made, but the response to the system is not known with certainty. A principal tool for quantitative modeling and analysis in this context is probability theory. Even though a system response is uncertain, specifying probabilities of the various possible responses will assist decision making.

Ex. 1. A Gambling Example

The Game of Craps

This is your first trip to Las Vegas and you are thinking about trying out the gaming tables. As a student, you don't have much money, so you're a little afraid of diving right in. After all, the minimum bet is $5, and it won't take many losses before your gambling budget of $100 is exhausted. You are looking forward to a good time, but the uncertainty and risk involved is unsettling.

The hotel has a special channel on the television that helps you learn the rules. You turn it on, and they are discussing the game of *Craps.* In this game, the player rolls a pair of dice and sums the numbers showing. A sum of 7 or 11 wins for the player, and a sum of 2, 3, or 12 loses. Any other number is called the point. The player then rolls the dice again. If she rolls the point number, she wins. If she throws a 7, she loses. Any other number requires another roll. The game continues until the gambler rolls a 7 or her point number.

Before going downstairs to the tables, you decide to do a little analysis. Perhaps that probability course you took in college will finally bear fruit. You are familiar with dice, so with a little thought you determine the probabilities of the possible results of the roll of a pair of dice and construct the table below.

Table 1. Probabilities for a throw of two dice

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
Cumulative	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	36/36

The activity that results in some uncertain outcome is an *experiment*, and some number observed or computed from the outcome is a *random variable*. The current experiment is throwing a pair of dice, and the random variable is the sum of the numbers facing up. The table showing the possible values of the random variable and their probabilities constitutes the model of the situation regarding the throwing of two dice. Games of chance are interesting examples, because experiments or logical arguments easily verify the probabilities describing the model.

The table row called labeled "sum" provides the possible observations of the random variable. The sum is a *discrete random variable*. The set of possible values is

$$\boldsymbol{X} = \{2, 3, \ldots , 12\}.$$

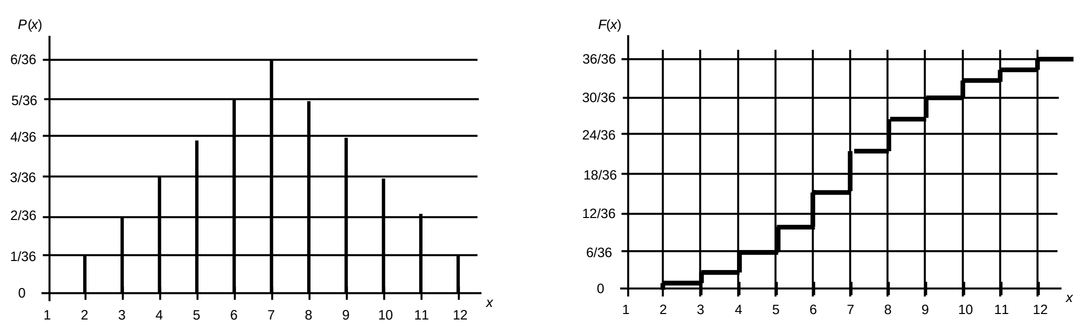
The row labeled probability shows the chance that a single throw results in each of the various possible values of the random variable. The collection of probabilities is *probability distribution function* (p.d.f.) of the random variable. Assign the notation $P_X(k)$ to represent the probability that the random variable X takes on the value k. Probabilities are always nonnegative, and their sum over all possible values of the random variable must be 1.

The *cumulative distribution function* (c.d.f.) describes the probability that the random variable is less than or equal to a specified value. The value of the cumulative distribution function at b is

$$F_X(b) = \sum_{k \le b} P_X(k)$$

We drop the subscript x on both P_X and F_X when there is no loss of clarity.

The distributions for this situation are graphed in Fig. 2. The p.d.f. has the value 0 at x =1, rises linearly to a peak at x= 7, and falls linearly to the value of 0 at x = 13. The function is called a *triangular distribution*. The c.d.f. shown in Fig. 2 has the typical pattern of all functions of this type, starting at 0 and rising until it reaches the value of 1. The function remains at 1 for all values greater than 12. For the discrete random variable, the p.d.f. has nonzero values only at the integers, while the c.d.f. is defined for all real values.



Probability distribution function Cumulative distribution function

Figure 2. Distribution functions for total on dice (Triangular Distribution)

Computing Probabilities of Events

With the probability distribution in hand, you easily compute the probability that you will win or lose on the first throw of the dice. Since the possible values are mutually exclusive, you simply add the probabilities to find the probability of win or lose.

$$P(\text{win}) = P(7) + P(11) = 6/36 + 2/36 = 0.222$$

$P(\text{lose}) = P(2) + P(3) + P(12) = 1/36 + 2/36 + 1/36 = 0.111$

The odds look good, with the probability of a win equal to twice that of a loss. You confidently head toward the tables. Unfortunately, the game does not go well, and you lose a good deal of your fortune. Perhaps your analysis stopped a little early. We continue this example later to find a better estimate of your chances of winning this game.

An *event* is a subset of the outcomes of an experiment. Event probabilities are computed by summing over the values of the random variable that make up the event. In the case of the dice, we identified the event of a "win" as the outcomes 7 and 11. The probability of a win is the sum of the probabilities for these two values of the random variable.

In many cases events are expressed as ranges in the random variable. In general, the probability that the discrete random variable falls between a and b is

$$P(a \le x \le b) = \sum_{k=a}^{b} P_x(k) = \sum_{k=0}^{b} P_X(k) - \sum_{k=0}^{a-1} P_X(k)$$

$$= F_X(b) - F_X(a-1) \text{ for } a \le b.$$

Range event probabilities are computed by summing over the values of the random variable that make up the event or differencing the cumulative distribution for the values defining the range.

For discrete random variables, the relation ≤ is different than the relation < since a nonzero probability is assigned to a specific value. Another useful expression comes from the fact that the total probability equals 1.

$$P(x \ge a) = 1 - F_X(a - 1).$$

To illustrate, several events for the dice example appear in Table 2. The random variable, x, is the sum of the two dice. The examples show how different phrases are used to describe ranges. The values of the cumulative distribution come from Table 1.

Table 2. Simple Range Events

Event	**Probability**
The sum on the dice is less than 7.	$P(x < 7) = P(x \le 6) = F_X(6) = 15/36$.
The sum is between 3 and 10 inclusive.	$P(3 \le x \le 10) = F_X(10) - F_X(2) = 32/36$.
The sum is more than 7.	$P(x > 7) = 1 - P(x \le 7) = 1 - F_X(7) = 15/36$.
The sum is at least 7.	$P(x \ge 7) = 1 - F_X(6) = 21/36$.
The sum is no more than 7.	$P(x \le 7) = F_X(7) = 21/36$.

Combinations of Events

Based on the results of the first night's play, you feel that you were a little rash considering only the probability of winning or losing on the first roll of the dice. In fact, most of your loses occurred when didn't roll a winning or losing number and were forced to roll for a point. You must roll for a point if the first roll is between 4 and 6 or between 8 and 10. You define the event $\boldsymbol{A}$ as the outcomes $\{4 \le x \le 6\}$ and the event $\boldsymbol{B}$ as the outcomes $\{8 \le x \le 10\}$. Using the rules of probability you determine that

$$P(\boldsymbol{A}) = F(6) - F(3) = 15/36 - 3/36 = 1/3;$$

$$P(\boldsymbol{B}) = F(10) - F(7) = 33/36 - 21/36 = 1/3.$$

The probability that you must roll a point is the probability that either event $\boldsymbol{A}$ or event $\boldsymbol{B}$ occurs. This is the union of these two events (written $\boldsymbol{A}\quad\boldsymbol{B}$). Again from your probability course you recall that since the events are mutually exclusive, the probability of the union of these two events is the sum of their probabilities.

$$P(\text{throw a point }) = P(\boldsymbol{A}\quad\boldsymbol{B}) = 1/3 + 1/3 = 2/3.$$

Together win, lose and throw a point represent all possibilities of the first roll, so it is not surprising that

$$P(\text{win}) + P(\text{lose}) + P(\text{throw a point}) = 0.222 + 0.111 + 0.667 = 1.$$

Events are sets of outcomes so we use set notation such as $\boldsymbol{A}$ and $\boldsymbol{B}$ to identify them. The union is two events, that is the outcomes in either $\boldsymbol{A}$ *or* $\boldsymbol{B}$, is written $\boldsymbol{A}\quad\boldsymbol{B}$. The intersection of two events, that is the set of outcomes in both events $\boldsymbol{A}$ *and* $\boldsymbol{B}$, is written $\boldsymbol{A}\cap\boldsymbol{B}$.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are mutually exclusive,
$P(\boldsymbol{A}\quad\boldsymbol{B}) = P(\boldsymbol{A}) + P(\boldsymbol{B})$.

When two events are mutually exclusive they have no outcomes in common, and the probability of their union is the sum of the two event probabilities as illustrated above.

Probability of the union of any two events is
$P(\boldsymbol{A}\quad\boldsymbol{B}) = P(\boldsymbol{A}) + P(\boldsymbol{B}) - P(\boldsymbol{A}\cap\boldsymbol{B})$.

When the events are not mutually exclusive, we have the more general expression where $P(\boldsymbol{A}\cap\boldsymbol{B})$, is the probability that both events occur. For example, let $\boldsymbol{A}$ = {roll less than 8} and $\boldsymbol{B}$ = {roll more than 6}. We compute

$$P(\boldsymbol{A}) = F(7) = 21/36,\ P(\boldsymbol{B}) = 1 - F(6) = 1 - 15/36 = 21/36.$$

To compute the probability of the union, we must first find the probability of the intersection event $P(\boldsymbol{A}\cap\boldsymbol{B}) = P(7) = 6/36$. Using the general expression for the union we find

$$P(\boldsymbol{A}) + P(\boldsymbol{B}) - P(\boldsymbol{A}\cap\boldsymbol{B}) = 21/36 + 21/36 - 6/36 = 1.$$

Indeed the union of these two events includes all possible outcomes.

> The probabilities of the intersection of independent events is
> $P(\boldsymbol{A} \cap \boldsymbol{B}) = P(\boldsymbol{A})\, P(\boldsymbol{B})$.

Two events are independent if the occurrence of one does not affect the probabilities associated with the other. As an example, we might be interested in the probability that two sixes are thrown for a pair of dice. The results of the two throws are independent, and we compute

> The probability of the union of independent events is
> $P(\boldsymbol{A} \quad \boldsymbol{B}) = P(\boldsymbol{A}) + P(\boldsymbol{B}) - P(\boldsymbol{A})\, P(\boldsymbol{B})$.

$$P(\{\text{six on first}\} \cap \{\text{six on second}\})$$

$$= P(\text{six on first})P(\text{six on second})$$

$$= (1/6)(1/6) = 1/36.$$

When events are not independent, they are related by conditional probabilities. Consider the events: $\boldsymbol{A}$ = {roll a 4 on the first play}, $\boldsymbol{B}$ = {win the game}. These events are not independent because the probability of winning depends on whether or not we roll a 4. We can however define the conditional probability of winning given that a 4 is rolled on the first play, written $P(\boldsymbol{B} \mid \boldsymbol{A})$. To compute this probability, we assume we have rolled a point of 4. On the second roll, the probability that we win, given that the game ends is

$$P(\text{roll a 4})/[P(\text{roll a 4}) + P(\text{roll a 7})] = 1/3.$$

> The general expression for the intersection probability is
> $P(\boldsymbol{A} \cap \boldsymbol{B}) = P(\boldsymbol{A})\, P(\boldsymbol{B} \mid \boldsymbol{A})$.

We can't say how many rolls it will take, but this probability remains the same throughout the remainder of the game, so we conclude that

$$P(\boldsymbol{B} \mid \boldsymbol{A}) = P(\text{win the game / roll a 4 on the first play}) = 1/3.$$

> For independent events,
> $P(\boldsymbol{B} \mid \boldsymbol{A}) = P(\boldsymbol{B})$ and $P(\boldsymbol{A} \mid \boldsymbol{B}) = P(\boldsymbol{A})$.

We use the general formula for computing the intersection probability

$$P(\boldsymbol{A} \cap \boldsymbol{B}) = (3/36)(1/3) = 1/36.$$

You now have enough tools to complete the analysis of the game. Consider an experiment that is a single complete play of the game. You define the following events:

$\boldsymbol{W}$: Win the game

$\boldsymbol{L}$: Lose the game

$\boldsymbol{W}_1$: Win on first roll. $P(\boldsymbol{W}_1) = 2/9$

$\boldsymbol{P}_{ike}$: Throw a point of k on the first roll, k = 4, 5, 6, 8, 9, 10.

The probabilities associated with the events are:

$P(\boldsymbol{P}_4) = 3/36$, $P(\boldsymbol{P}_5) = 4/36$, $P(\boldsymbol{P}_6) = 5/36$, $P(\boldsymbol{P}_8) = 5/36$,

$P(\boldsymbol{P}_9) = 4/36$, $P(\boldsymbol{P}_{10}) = 3/36$.

The conditional probabilities, $P(\ \boldsymbol{W}\ |\boldsymbol{P}_{ike})$, probability of a win given a point of k on the first roll.

$P(\boldsymbol{W}\ |\boldsymbol{P}_4) = 1/3$, $P(\boldsymbol{W}\ |\boldsymbol{P}_5) = 4/10$, $P(\boldsymbol{W}\ |\boldsymbol{P}_6) = 5/11$, $P(\boldsymbol{W}\ |\boldsymbol{P}_8) = 5/11$, $P(\boldsymbol{W}\ |\boldsymbol{P}_9) = 4/10$, $P(\boldsymbol{W}\ |\boldsymbol{P}_{10}) = 1/3$.

The event of winning the game combines all possible winning combinations of events.

$$\boldsymbol{W} = \boldsymbol{W}_1 \quad (\boldsymbol{P}_4 \cap \boldsymbol{W}_4) \quad (\boldsymbol{P}_5 \cap \boldsymbol{W}_5) \quad (\boldsymbol{P}_6 \cap \boldsymbol{W}_6) \quad (\boldsymbol{P}_8 \cap \boldsymbol{W}_8)$$

$$(\boldsymbol{P}_9 \cap \boldsymbol{W}_9) \quad (\boldsymbol{P}_{10} \cap \boldsymbol{W}_{10})$$

$$P(\boldsymbol{W}) = P(\boldsymbol{W}_1) + P(\boldsymbol{P}_4)\,P(\boldsymbol{W}|\boldsymbol{P}_4) + P(\boldsymbol{P}_5)\,P(\boldsymbol{W}|\boldsymbol{P}_5) + P(\boldsymbol{P}_6)\,P(\boldsymbol{W}|\boldsymbol{P}_6)$$

$$+ P(\boldsymbol{P}_8)\,P(\boldsymbol{W}|\boldsymbol{P}_8) + P(\boldsymbol{P}_9)\,P(\boldsymbol{W}|\boldsymbol{P}_9) + P(\boldsymbol{P}_{10})\,P(\boldsymbol{W}|\boldsymbol{P}_{10})$$

$P(\boldsymbol{W}) = 0.493$.

If you don’t win you lose, so $P(\boldsymbol{L}) = 1 - P(\boldsymbol{W}) = 0.507$.

Your chances of winning on any one play are slightly less than 50%. You now know that the casino has a better chance of winning the game than you, and that if you play long enough you will lose all your money. If this were an economic decision, your best bet is not to play the game. If you do decide to play, you go to the tables armed with the knowledge of your chances of winning and losing. The probability model has changed the situation to one of risk rather than uncertainty. You can use the probability model to answer many other questions about this game of chance.

Ex. 2. An Example from Manufacturing

You are an machine operator and have recently been promoted to be in charge of two identical machines in a production line. You just received your first performance review from the supervisor. The report is not good and it ends with the comment,

"Your station fails in both efficiency and work-in-process. You are inconsistent in your work habits. At times you sit idle and at other times waiting work is overflowing from your station. A work study has determined that you have sufficient capacity to do all the work assigned. It's hard to imagine how you can fail to complete all your work, have idle time and excess work-in-process. If you don't improve, we'll have to reevaluate your promotion."

You throw down the review in anger, blaming the station that comes before yours as not providing the work on a regular basis. You also blame the maintenance department for not keeping your machines in good shape. They often require frequent adjustments and no two jobs take the same time. You also complain about the production manager who takes work away from you and gives it to another station whenever the number waiting exceeds 6 jobs. After you cool down you pledge to yourself to try to improve. You really need this job and the extra pay it provides.

You visit the industrial engineering department and ask for some ideas to improve your efficiency. The engineer listens to your problem and points out that perhaps the situation is not your fault. He explains that variability in the arrival process and machine operation times can result in both queues and idleness. Using data from the work study he determines that you are assigned an average of 50 jobs per week and that each job takes an average of 1.4 hours for a total work load of 70 hours per week. With two machines working 40 hours per week, you have a production capacity of 80 hours, sufficient time to do the work assigned. As a first attempt at modeling your station, the engineer uses a queueing analysis program and finds the following probability distribution for the number of jobs that are at your station.

Number	0	1	2	3	4	5	6	7	8
Probability	0.098	0.172	0.15	0.131	0.115	0.101	0.088	0.077	0.067
Cumulative	0.098	0.27	0.42	0.552	0.667	0.767	0.855	0.933	1.

The table does give some interesting insights. The model predicts that you are idle almost 10% of the time (when the number in the system is 0). About 7% of the jobs entering the station will find all six queue positions full (when the number in the system is 8). Since these jobs are sent to another operator, the number of jobs actually processed in your station is 50*(1 - 0.067) = 46.63. Between three and four jobs per week are taken away from you because of congestion at your station. The remaining jobs require an average of 46.63*1.4 = 65.3 hours per week. Since your machines have 80 hours of capacity, their efficiency is 65.3/80 = 82%. The work-in-process (WIP) is the expected number of jobs in the station. Using probability theory you calculate the expected value for the number of jobs as

$$\mu = \sum_{k=0}^{8} kP_x(k) = 0*0.098 + 1*0.172 + 2*0.15 \ldots + 8*0.067 = 3.438.$$

You are not very encouraged by these results. They clearly support your supervisor's opinion that you are simultaneously inefficient, lazy, and generally not doing your job. The kindly engineer consoles you that the results are the inevitable cause of variability. You can do nothing about the situation unless you can reduce the variability of the arrivals of jobs to your station, the variability of the processing times or both. If these factors are beyond your realm of control, you better try to convince your supervisor that the problem is not your fault and that he should look elsewhere for solutions.

Both the gambling and manufacturing examples illustrate the use of probability models to learn more about situations. The gambling example illustrates a case where the underlying causes of variability are well known and the analysis yields accurate estimates about the probabilities of winning and losing. The model for the manufacturing example rests on arguable assumptions and the results may not be accurate. The model is valuable however, in that it illustrates an aspect of the situation often not apparent to decision makers. The results indicate that random variability is the source of the problem and should stimulate the search for new solutions.