Game Theory

Most of operations research deals with problems that are very complex or that in some way are affected by random events. When the problems are deterministic, a large variety of them can be solved, at least conceptually, with the methods discussed in the first part of the book regardless of their complexity. Randomness clouds the definition of optimality and makes the decision process that much more difficult due to our inability to predict outcomes with certainty. The methods of operations research provide the analytic tools that give additional information to the decision maker, and in many cases allow optimal decisions to be prescribed. In Chapter 23, we considered decision analysis, a process that leads to policies that are optimal even in the presence of random events. With decision analysis we are assuming that nature, our "opponent," is fair or unbiased. The probability distribution for a random event may be affected by previous events and decisions, but the event is selected by a fair lottery according to the given probability distribution.

The decision problems we often involve another individual who may be an antagonistic opponent. We must make decisions knowing that the result will be governed in part by the actions of a competitor. Husbands and wives must deal with each, parents with their children, businesses with their competitors, military commanders with their enemies, and game players with their opponents. The part of operations research that addresses this kind of situation is called *game theory*. Although clearly applicable to games as the name implies, it is appropriate in a wide variety of contexts in which one must make a decision whose outcome will be determined by the actions of one or more individuals.

We introduce the subject in this chapter by considering a very simple game with two opposing players. It is called a zero-sum game in which a gain to one player is a loss to the other. All information concerning the game is known to both players; i.e., there is complete information. The players are said to be rational and intelligent. A rational person is one who acts in such a way as to maximize his or her expected payoff or utility as economists would say. An intelligent person is one who can deduce what his or her opponent will do when acting rationally.

24.1 Basic Models

With a game theory model, we provide a mathematical description of a social situation in which two or more individuals, or players, interact. Such a broad scope allows many kinds of models. There may be two players or many. The players may be competitive or cooperative. With more than two players there may be collusion among subsets of the players. Games may involve several sequential steps or one step for each player. Competitive situations may be repeated or be faced only once. Information concerning the rules of engagement and the payoffs may be known to all players or imperfectly known to some. In an introductory discussion, we can only touch on this important and valuable subject. We restrict attention to the simplest model -- the two-person, zero-sum game.

The Two-Person, Zero-Sum Game

Consider a competitive situation with two players and arbitrarily assume that player I is a woman and player II is a man. The game is specified by the sets of strategies available to the two players and the payoff matrix. The set of strategies for player I is indexed from 1 through m. The set of strategies for player II is indexed from 1 through n. The payoff matrix (see Table 1) specifies the gain or profit to player I for every strategy pair (i, j).

Table 1. Payoff Matrix

		Player II			
		1	2	…	n
Player I	1	p_{11}	p_{12}	$\cdots$	p_{1n}
	2	p_{21}	p_{22}	$\cdots$	p_{2n}
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

The two players select their strategies simultaneously, and when player I uses strategy i and player II uses strategy j, player I receives the payoff p_{ij} from player II. A positive number is a gain for player I and a negative number is a loss (a gain for player II). A gain to one player is a loss to the other, thus providing the zero-sum feature. The payoff obtained when the two players select their strategies is the *value* of the game. Each player know all strategies available to the other, and they both agree on the payoff matrix.

We will see that this is not a very interesting game because the optimal strategies for both players can be determined in advance and neither player can improve his or her position by changing the prescribed strategy. Solutions are either pure or mixed strategies. With a pure strategy a player chooses only one strategy in a play of the game. In a mixed strategy, a player chooses one of several strategies according some probability distribution.

Tic-Tac-Toe

To illustrate, consider this very simple example of a two-person game. As most readers will know, the game is played on the diagram shown in Fig. 1a. The first player begins by entering an X in one of the nine available spaces. The second player places a O in one of the eight remaining places. The two players continue taking turns in this fashion (first X and then O) until either all squares are filled, it becomes clear that one player has won, or it is clear that neither player can win. The game is won if three of one player's symbols are arranged in a straight line either horizontally, vertically or diagonally. Figure 1 shows an example where the sequence of plays b through h results in a win for X. The game frequently ends in a tie with no player winning, as shown in Fig. 1i.

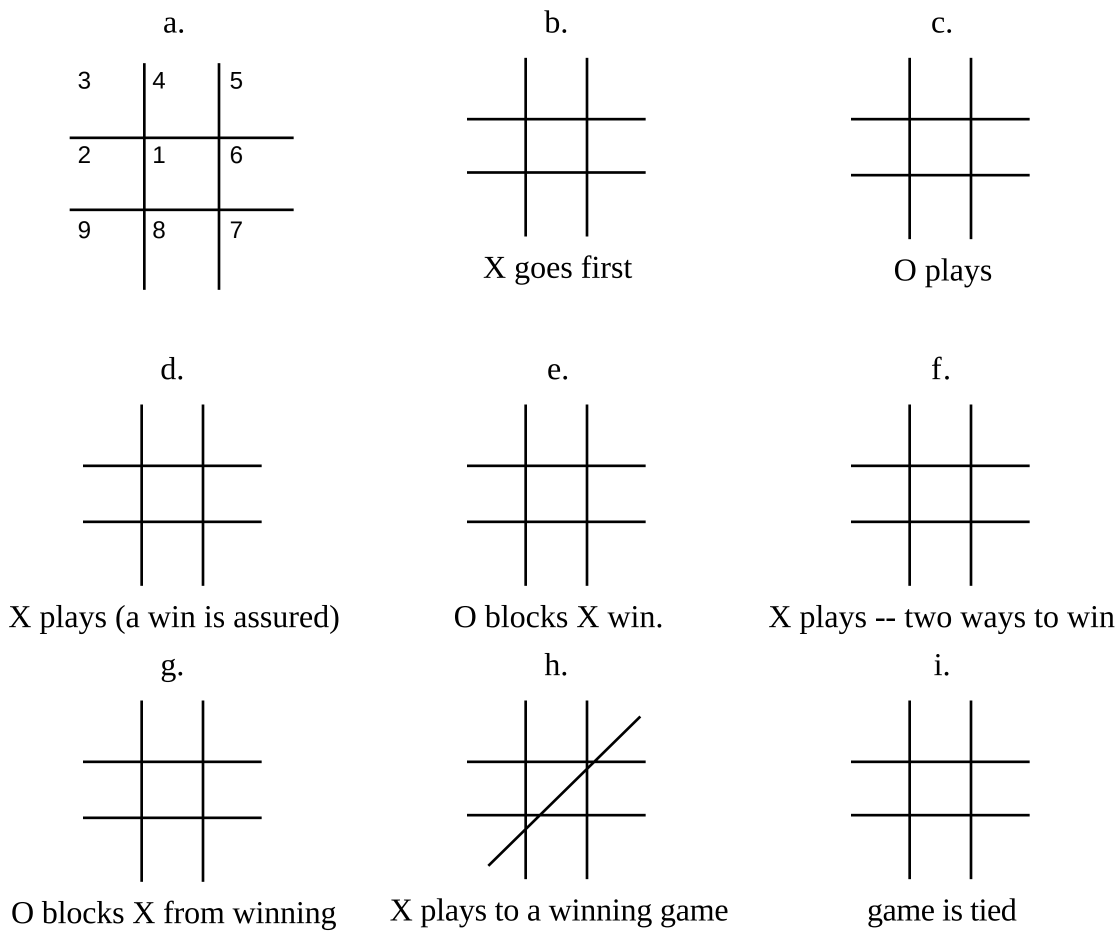
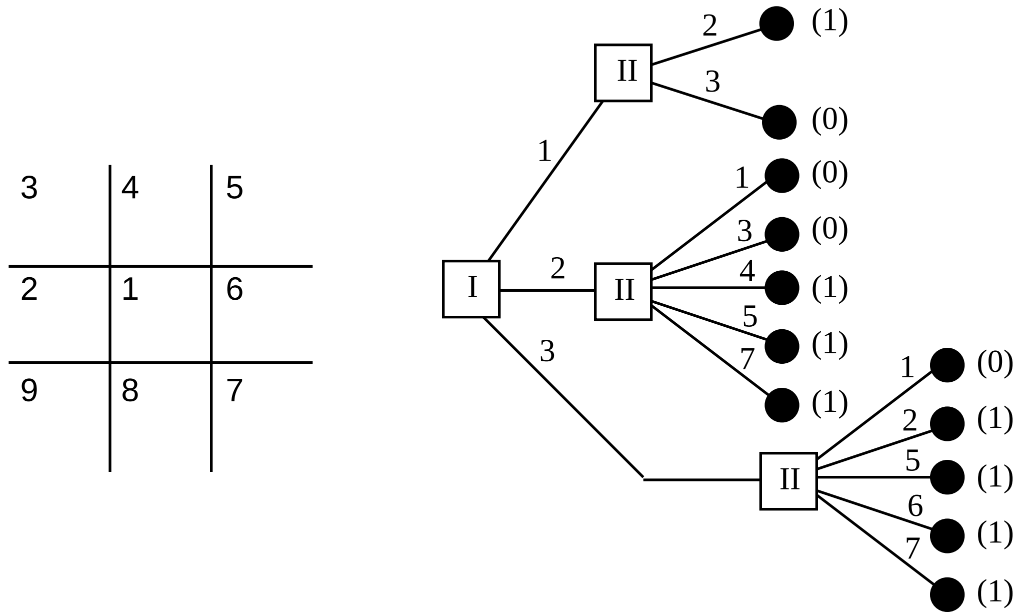


Figure 1. The tic-tac-toe game

To analyze this game, one might use a decision tree as described in Chapter 23. As all experienced players of know, the result of the game is completely determined after the first X and the first O have been played, assuming neither player makes a mistake. Representing a win for X as the value 1, a tie with a 0, and a win for O as a –1, Fig. 2 shows the decision tree based on the first two plays. Because the game is symmetric it is only necessary to consider squares 1, 2 and 3 for player I and a reduced number of squares for player II. Any moves left out are symmetric to one of those shown. Because player I goes first, her three possible plays are in square 1, 2 and 3. Given the choice of player I, player II has the several remaining squares as a possible response.



a. Game board b. Decision tree

Figure 2. The decision tree showing the possible results of tic-tac-toe

The results of the game given the first two plays are shown in the figure. The game shown in Fig. 1b - Fig. 1h follows the top path of the decision tree where player I uses square 1, the center, and player II follows with square 2. The ultimate result is a win for player I. The complete game in Fig 1i is the result obtained when player I starts with square 2 and player II counters with square 3. The game is always a tie when the successive plays are made intelligently.

The tree shown in Fig. 2 is a model of the game and is called the extensive form. It depicts the sequential nature of the game. Even for this simple situation, the extensive form can be very large. If we had not recognized the symmetry of the game and the fact that only the first two moves are important, the tree would be impossible to show on a single page.

The Strategic Form

Game theory uses an alternative model called the strategic form which represents the game as a matrix. The assumption of the strategic form is that both players select strategies before the game is played and simply act out those strategies in turn. A strategy must describe a plan of action for every possible situation.

As a first attempt, we might define a strategy for player I as her first play and the strategy for player II as his response. This definition does not work for player II. It does not prescribe what player II should do if player I uses the square that he was intending to use.

To construct the strategic form of the tic-tac-toe game, we define the following strategies regarding the first moves for the two players.

Player I: Select one of the nine squares on the game board.

Player II: Select one of the nine squares on the game board. If player I uses the selected square,

- put an O in square 3, 5, 7, or 9 if an X is in square 1 (center)
- put an O in cell 1 If an X is in cell *j*.

These are complete strategies. The players can select them before the game begins and follow them through the first two plays. The strategic form of the game is represented in Table 2. The entries in the table are the values of the game for every possible selection of strategies.

Table 2. Strategic Form of the Tic-Tac-Toe Game

		Strategy for Player II, O								
		1	2	3	4	5	6	7	8	9
	1	0	1	0	1	0	1	0	1	0
	2	0	0	0	1	1	0	1	1	0
	3	0	1	0	1	1	1	1	1	1
Strategy for	4	0	1	0	0	0	1	1	0	1
Player I, X	5	0	1	1	1	0	1	1	1	1
	6	0	0	1	1	0	0	0	1	1
	7	0	1	1	1	1	1	0	1	1
	8	0	1	1	0	1	1	0	0	0
	9	0	1	1	1	1	1	1	1	0

Conceptually, every game that can be described by an extensive form like that in Fig. 2 has an equivalent strategic form similar to the one shown in Table 2. For convenience, we will use the strategic form for further analysis. It is equivalent to the payoff matrix introduced earlier. The payoff matrix together with the descriptions of the strategies constitute the model for the two-person, zero-sum game.

24.2 Solution Methods

We solve the game by prescribing the optimal strategies each player should adopt. The game is played by the players following the strategies. Solutions may involve pure strategies, in which each player uses only one play, or mixed strategies, in which plays are selected randomly according to some probability distribution. The *value of the game* is the payoff obtained when both players follow their optimal strategies.

Dominated Strategies

A *dominated strategy* is a strategy that yields a payoff for one of the players that is less than or equal to the payoff for some other strategy for all actions of the opponent. For player I, strategy i is dominated by strategy k if

$$p_{ij} \le p_{kj} \text{ for } j = 1, \ldots, n \tag{1}$$

In other words i is dominated by k if every element of row i in the payoff matrix is less than or equal to every corresponding element of row k.

For player II, a strategy is dominated if every element of a column is greater than or equal to every corresponding element of some other column. Strategy j is dominated by strategy k if

$$p_{ij} \ge p_{ik} \text{ for } i = 1, \ldots, m. \tag{2}$$

It should be clear that dominated strategies will not be used in the solution of the game. If a dominated strategy is used, a better solution is always obtained by replacing it with the dominating strategy. The first step in the solution process is to find and eliminate the dominated strategies from the game.

Using the tic-tac-toe payoff matrix in Table 1, we observe that column 1 is smaller than every other column. Strategy 1 for player II dominates every other strategy. We can, therefore, eliminate all other columns. The resulting matrix has one column with all zeroes. For player I, all strategies are equal, and the game will always end in a tie. The optimum strategy is

Player I: Select one of the nine squares of the game board

Player II: Always use the center square unless player I chooses it

If player I uses the center square, put an O in square 3, 5, 7, or 9.

The dominance argument shows what many have discovered by trial and error, tic-tac-toe is not a very interesting game for intelligent players. One can win only if one of the players makes a mistake.

We use the more interesting hypothetical payoff matrix in Table 3 for another illustration. To describe the sequence of operations in a solution we call row i and column j of the current payoff matrix $\mathbf{r}_i$ and $\mathbf{c}_j$ respectively. We say *current* because the payoff matrix changes with the elimination of strategies. Then

- row i is dominated by row k if $\mathbf{r}_i \le \mathbf{r}_k$, and

- column j is dominated by row k if $\mathbf{c}_j \geq \mathbf{c}_k$.

Table 3. Payoff Matrix with Dominating Strategies

		Player II			
		1	2	3	4
Player I	1	5	4	–3	–2
	2	2	–1	–2	0
	3	2	4	–1	0
	4	1	3	2	1

We start with player I and find any dominated strategies. To do this we must compare every pair of rows to see if the condition identified by Equation (1) is satisfied. Dominated strategies are eliminated by deleting the associated rows from the payoff matrix. For player II, we find and eliminate dominated strategies by comparing all pairs of columns to see if the Equation (2) is satisfied. The associated dominated columns are then deleted. Since deleting rows and columns may uncover new dominated strategies, we continue the process until no dominated strategies remain for either player. If only one strategy remains for both players, these strategies are the solution to the problem.

We solve the example problem by first noting that

$$\mathbf{r}_2 \leq \mathbf{r}_3,\ \mathbf{c}_1 \geq \mathbf{c}_4, \text{ and } \mathbf{c}_2 \geq \mathbf{c}_3.$$

Strategy 2 is dominated for player I and strategies 1 and 2 are dominated for player II. Deleting the associated rows and columns of the dominated strategies we obtain a new payoff matrix.

		Player II	
		3	4
Player I	1	–3	–2
	3	–1	0
	4	2	1

Now

$$\mathbf{r}_1 \leq \mathbf{r}_3,\ \mathbf{r}_3 \leq \mathbf{r}_4$$

and the new matrix is

		Player II	
		3	4
Player I	4	2	1

When only one row (or column) remains, it is always possible to reduce the number of columns (or rows) to one by dominance. We then find the solution to the game with both players using strategy 4. The value of the game is equal to 1.

Say the game was sequential rather than both players selecting their plays simultaneously. Table 4 gives the payoff matrix when player I goes first. One might suspect player II has the advantage since his play is obvious given the selection of player 1.

Table 4. Payoff Matrix when Player I Goes First

		Player II 1	2	3	4	Best for II	Value
	1	5	4	–3	–2	3	–3
Player I	2	2	–1	–2	0	3	–2
	3	2	4	–1	0	3	–1
	4	1	3	2	1	1 or 4	1

When player I selects any strategy except 4, player II will do better. But knowing the payoff matrix and assuming player II is rational, why should she use any strategy except 4? The only rational solution is for player I to use strategy 4 and player II to counter with strategy 1 or 4.

Saddle Point Solutions

Using dominance, a strategy is eliminated if it is inferior to (or at best no better than) some other strategy in all respects. This uses what most would consider as a reasonable characteristic of rationality, it is better to receive a larger payoff than a smaller payoff. What if dominance cannot be used to eliminate all but one strategy as in the matrix of Table 5?

Table 5. Payoff Matrix for Saddle Point Example

		Player II 1	2	3
	1	5	1	0
Player I	2	0	2	7
	3	4	3	4

Consider the decision problem of player I. In the worst case assume that her opponent knows her decision. If she chooses strategy 1, he will choose strategy 3 for a value of 0. If she chooses strategy 2, he will choose strategy 1 for a value of 0. If she chooses strategy 3, he will choose strategy 2 for a value of 3. This information is obtained by selecting the minimum value for each row and the associated best strategy for player II.

Using a similar analysis for player II. If he chooses strategy 1, she will choose strategy 1 for a value of 5. If he chooses strategy 2, she will choose strategy 3 for a value of 3. If he chooses strategy 3, she will choose strategy 2 for a value of 7. This information is obtained by selecting the maximum value for each column and the associated best strategy for player I. Table 6 summarizes the information for both players.

Table 6. Payoff Matrix with Saddle Point Strategies

		Player II 1	2	3	Minimum of row	Strategy for II
	1	5	1	0	0	3
Player I	2	0	2	7	0	1
	3	4	3	4	3	2
Maximum for column		5	3	7		
Strategy for I		1	3	2		

If both players assume their opponents are rational and intelligent, the obvious conclusion is that player I should select strategy 3 and player II should select strategy 2, and the value of the game is 3. When player I selects strategy 3, any other choice beside strategy 2 for player II will result in a greater loss for player II. When player II selects strategy 2, any other choice except strategy 3 for player I results in a smaller return for player I.

The cell (3, 2) is called the saddle point of the game. The two players are using the minimax criterion for strategy selection. Not every game has a saddle point, but when it does the saddle point is the solution of the game.

In general terms, the pure minimax strategy for player I is the strategy that maximizes her minimum gain. The payoff for this strategy is v_L where

$$v_L = \text{Max}_{i = 1,\ldots,m}\{\text{Min}_{j = 1,\ldots,n}\, p_{ij}\} = \text{Max}\{0, 0, 3\} = 3$$

The row that determines the maximum is the pure minimax strategy for player I. This is row 3 for the example. The pure minimax strategy for player II is the strategy that minimizes the maximum gain for player I. The payoff for this strategy is v_U where

$$v_U = \text{Min}_{j = 1,\ldots,n}\, \{\text{Max}_{i = 1,\ldots,m}\, p_{ij}\} = \text{Min}\{5, 3, 4\} = 3$$

The column that obtains the minimum is his pure minimax strategy for player II. This is column 2 for the example.

The quantities v_L and v_U defined above are called the lower and upper value of the game, respectively. When these two values are the same, the common result, $v = v_L = v_U$ is called the *value of the game*. When the same element of the payoff matrix determines the minimax strategies for both players, the upper and lower values are equal, and that element is called the *saddle point*. A *stable game* is a game with a saddle point. In this game, both players can adopt the pure minimax strategy and cannot improve their positions by moving to any other strategy.

Because both players are intelligent, they both know the payoff matrix. Because they are both rational, the only possible solution for this game is the saddle point.

An *unstable game* is a game with no saddle point. In this case upper and lower values are not equal. A player cannot adopt the pure minimax strategy without providing the opportunity for the opponent to gain an improved return. The next section describes how this situation is handled with mixed strategies.

Mixed Strategies

Consider the payoff matrix in Table 7. We see for this matrix

$$v_L = \text{Max}_{rows}\{\text{Min}_{columns}\ p_{ij}\} = \text{Max}_{rows}\{-2, 1\} = 1$$

The pure maximin strategy for player I is strategy 2.

$$v_U = \text{Min}_{columns}\ \{\text{Max}_{rows}\ p_{ij}\} = \text{Min}\{4, 4, 6, 7\} = 4.$$

The pure minimax strategy for player II is strategy 1. Thus, the lower and upper bounds on the value of the game are not the same so no saddle point exists. If both players used their pure minimax strategy, the payoff for the game would be 2; however, this is not the solution. Rather the solution is one in which both players use a mixed strategy.

Table 7. Payoff Matrix for Example Using Mixed Strategies

		Player II				Minimum	Strategy
		1	2	3	4	of row	for II
Player I	1	4	0	6	–2	–2	4
	2	2	6	1	7	**1**	**3**
Maximum for column		**4**	6	6	7		
Strategy for I		**1**	2	1	2		

A *mixed strategy* is procedure for playing the game by which each player chooses the strategy using a discrete probability distribution. For player I, the probabilities are $(x_1, x_2, \ldots, x_m)$, where x_i is the probability that strategy i will be chosen. For player II, the probabilities are $(y_1, y_2, \ldots, y_n)$, where y_j is the probability that strategy j will be chosen. Both sets of probabilities must sum to 1. The expected value of the payoff of the game when the players use a mixed strategy.

$$\text{Expected payoff} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j \ .$$

$$\sum_{i=1}^{m} x_i = 1, \quad \sum_{j=1}^{n} y_j = 1$$

Each player will use the mixed strategy that will minimize his or her maximum expected loss. In terms of the payoff, player I will adopt a mixed strategy that will maximize the minimum expected payoff, and player II will adopt a mixed strategy that will minimize the maximum expected payoff . If both players follow their minimax strategies, the value of the game, v, is equal to the expected payoff.

To describe the logic of the mixed strategy, consider the problem of player I when she is selecting the probabilities $(x_1, x_2, \ldots, x_m)$. She knows that the expected payoff of the game to player I when player II uses pure strategy j is

$$\sum_{i=1}^{m} p_{ij} x_i \quad \text{for } j = 1,\ldots,n$$

As an intelligent and rational person, player II selects a strategy that will assure that the value of the game for player I, v_{I}, is less than or equal to the expected value for any pure strategy j. Then we can write for each j the constraint

$$\sum_{i=1}^{m} p_{ij} x_i \geq v_{\mathrm{I}} \quad \text{for } j = 1,\ldots,n.$$

Now player I will select values for the probabilities that will maximize her return. Together this logic leads to the linear programming solution for player I's strategy.

$$\text{Maximize } v_{\mathrm{I}}$$

$$\text{subject to} \quad \sum_{i=1}^{m} p_{ij} x_i \geq v_{\mathrm{I}}, \quad j = 1,\ldots,n \tag{3a}$$

$$\sum_{i=1}^{m} x_i = 1 \tag{3b}$$

$$x_i \geq 0, \quad i = 1,\ldots,m \tag{3c}$$

The last two constraint sets assure that the decision variables x_i define an acceptable probability distribution.

It is not hard to show that the dual of this model solves the selection problem for player II. Assigning the dual variables y_j to the constraints in the set (3a), we obtain the dual problem.

$$\text{Minimize } v_{\mathrm{II}}$$

$$\text{subject to} \quad \sum_{j=1}^{n} p_{ij} y_j \leq v_{\mathrm{II}}, \quad i = 1,\ldots,m. \tag{4a}$$

$$\sum_{j=1}^{n} y_j = 1 \tag{4b}$$

$$y_j \geq 0, \quad j = 1,\ldots,n \tag{4c}$$

Solving either one of these linear programs yields the optimal solution for one player. The dual values provide the solution for the other player.

We repeat the payoff for the example matrix in Table 8 so that we can construct the linear programming model.

Table 8. Payoff Matrix in Table 7

		Player II			
		1	2	3	4
Player I	1	4	0	6	–2
	2	2	6	1	7

The linear programming model for player I is

$$\begin{aligned}
&\text{Maximize } v \\
&\text{subject to } \quad 4x_1 + 2x_2 \geq v \\
&\qquad\qquad\qquad 6x_2 \geq v \\
&\qquad\qquad\qquad 6x_1 + 1x_2 \geq v \\
&\qquad\qquad\qquad -2x_1 + 7x_2 \geq v \\
&\qquad\qquad\qquad x_1 + x_2 = 1 \\
&\qquad\qquad\qquad x_1 \geq 0,\ x_2 \geq 0
\end{aligned}$$

and the dual model for player II is

$$\begin{aligned}
&\text{Minimize } v \\
&\text{subject to } \quad 4y_1 + 6y_3 - 2y_4 \leq v \\
&\qquad\qquad\qquad 2y_1 + 6y_2 + 1y_3 + 7y_4 \leq v \\
&\qquad\qquad\qquad y_1 + y_2 + y_3 + y_4 = 1 \\
&\qquad\qquad\qquad y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0
\end{aligned}$$

Solving, we get the solution

$$(x_1, x_2) = (5/11, 6/11) \text{ with } v = 32/11$$

for the primal problem and

$$(y_1, y_2, y_3, y_4) = (9/11, 0, 0, 2/11)$$

for the dual problem.

A Modified Tac-Tac-Toe Game

As another example we modify the tic-tac-toe game so that the players select their opening moves before the game begins. To give player II a better chance, we say that player II wins if both players select the same square for the initial move. Otherwise we proceed to play the game as before. The new payoff matrix is shown as Table 9.

Table 9. Strategic Form of the Tic-Tac-Toe Game when X goes first.

		Player II: First play for O									
		1	2	3	4	5	6	7	8	9	Min
	1	–1	1	0	1	0	1	0	1	0	–1
	2	0	–1	0	1	1	0	1	1	0	–1
	3	0	1	–1	1	1	1	1	1	1	–1
Player I:	4	0	1	0	–1	0	1	1	0	1	–1
First play	5	0	1	1	1	–1	1	1	1	1	–1
for X	6	0	0	1	1	0	–1	0	1	1	–1
	7	0	1	1	1	1	1	–1	1	1	–1
	8	0	1	1	0	1	1	0	–1	0	–1
	9	0	1	1	1	1	1	1	1	–1	–1
	Max	0	1	1	1	1	1	1	1	1	

We notice from this table that

$$v_L = -1 \text{ and } v_U = 0.$$

This game does not have a saddle point solution. Using the theory of this section, we can construct a linear programming model to find the optimal strategies:

Player I (X player): Use a mixed strategy with $x_2 = 0.5$, $x_3 = 0.25$ and $x_5 = 0.25$

Player II (O player): Use pure strategy 1

The expected value of the game is 0. Although player II can adopt a pure strategy, it is important that player I adopt a mixed strategy, not including the action of putting an X in the center. There are a number of alternative optimal solutions for player II, but all involve a mixed strategy. Any pure strategy adopted by player I will result in a sure win for player II.

24.3 What Makes Games Interesting

All the examples we have described have simple solutions. The optimal strategies for both players are determined entirely by the payoff matrix and may be described before the game is played. We might rank the three cases in terms of simplicity of analysis and play.

- The solution is determined by dominance. Here all but one strategy for each player is determined by observation and elimination. One might suggest that the eliminated strategies were redundant in the first place since they add nothing to the opportunities available to the players. The game really has only one rational strategy for each player.
- Minimax strategies determine a stable solution. Here the strategies are beneficial if the opponent makes the wrong play. Intelligent and rational players, however, will always select the saddle point strategy. The game could be played sequentially with either player going first and revealing his or her selection. The opponent can't improve the result with the added information.
- Minimax strategies do not determine a stable solution. Here the players must select a strategy from several possibilities using a probability distribution. The results of a single play are uncertain but the expected value is fixed. It is important that the first player's selection be hidden because the second player could take advantage of the result.

Games of this type are really not very interesting because once the analysis is complete, there is no point in playing the game. For a variety of reasons, competitive situations found in practice do not have this characteristic.

- There are too many strategies to enumerate. This is typical of board games such as chess. A strategy must describe all possible actions in all possible situations. For chess as for most interesting games, the number is simply to large to imagine or evaluate.
- Players are not always intelligent or rational. Once we assume this for one player, the other player might try to take advantage of this circumstance.
- There are often more than two players. Now we have the possibilities for cooperation or collusion between players.
- Real-life situations are not zero-sum games. In practice, it may be more appropriate to specify for each strategy the utility of the results for both players. In such cases, cooperation may be a better policy than competition.

The subject of game theory has something to say about each of these situations, but the solutions are not simple enough to describe in this introductory discussion. Certainly, continued study of this subject is justified since so many real decision situations involve the interaction two or more individuals.

24.4 Exercises

For Exercises 1 through 4 use the given payoff matrix for player I to find the optimal minimax strategies for both players.

1.

		Player II			
		1	2	3	4
Player I	1	–1	0	4	–1
	2	5	1	3	1
	3	–2	0	–1	4
	4	–3	–2	4	5

2.

		Player II			
		1	2	3	4
Player I	1	–2	3	5	2
	2	5	–1	–2	0

3.

		Player II			
		1	2	3	4
Player I	1	5	3	6	1
	2	4	2	2	4
	3	3	4	3	5

4.

		Player II				
		1	2	3	4	5
Player I	1	–7	–3	0	4	6
	2	1	–5	–2	0	2
	3	5	0	–5	–8	–6
	4	–3	–2	4	0	2
	5	–5	–2	1	5	0

5. Two companies, A and B, are bidding on a contract for the government. Three bids are possible: low, medium, and high. The lowest bid placed by the two companies will be chosen, but if the bids are equal, the government will choose company A over B. Both companies have the same data regarding the cost of the project. If a company wins with the low bid, the costs will equal the income and no profit is made. If a company wins with a medium bid a profit of 5 is realized. If a company wins with a high bid, a profit of 10 is earned. The table below shows the payoff matrix for company A. If the company losses the bid to its opponent, this is counted as a loss

with respect to the relative position of the two companies. The bids are sealed and the companies have no information about each other except that they both use a minimax policy. What is the optimum bidding policy for the two companies? Who will win the contract? What profit will be made?

		Company B strategy		
		Low	Med.	High
Company A strategy	Low	0	0	0
	Med.	0	5	5
	High	0	–5	10

6. At a particular point in a football game, the offensive team and the defensive team must select a strategy for the next play. Both teams have the same statistical information that predicts the expected yardage to be gained for various combinations of offensive and defensive strategies. The information is given in the table, which shows yards gained for the offensive team for every combination. Determine the minimax policy for both teams. What is the expected gain in yardage?

Payoff Matrix for Offense (yards gained)

		Defensive strategy				
		1	2	3	4	5
Offensive strategy	1	–2	0	2	4	5
	2	–2	–4	0	5	10
	3	10	0	–3	5	10
	4	20	15	5	–5	0
	5	30	20	8	0	–10

7 In the previous exercise, if the offensive team could add 5 to any individual cell in the table, where should the increment be added to most increase the expected payoff? How will the strategies and the value of the game be changed?

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