# Inventory Theory

Inventories are materials stored, waiting for processing, or experiencing processing. They are ubiquitous throughout all sectors of the economy. Observation of almost any company balance sheet, for example, reveals that a significant portion of its assets comprises inventories of raw materials, components and subassemblies within the production process, and finished goods. Most managers don't like inventories because they are like money placed in a drawer, assets tied up in investments that are not producing any return and, in fact, incurring a borrowing cost. They also incur costs for the care of the stored material and are subject to spoilage and obsolescence. In the last two decades their have been a spate of programs developed by industry, all aimed at reducing inventory levels and increasing efficiency on the shop floor. Some of the most popular are conwip, kanban, just-in-time manufacturing, lean manufacturing, and flexible manufacturing.

Nevertheless, in spite of the bad features associated with inventories, they do have positive purposes. Raw material inventories provide a stable source of input required for production. A large inventory requires fewer replenishments and may reduce ordering costs because of economies of scale. In-process inventories reduce the impacts of the variability of the production rates in a plant and protect against failures in the processes. Final goods inventories provide for better customer service. The variety and easy availability of the product is an important marketing consideration. There are other kinds of inventories, including spare parts inventories for maintenance and excess capacity built into facilities to take advantage of the economies of scale of construction.

Because of their practical and economic importance, the subject of inventory control is a major consideration in many situations. Questions must be constantly answered as to when and how much raw material should be ordered, when a production order should be released to the plant, what level of safety stock should be maintained at a retail outlet, or how in-process inventory is to be maintained in a production process. These questions are amenable to quantitative analysis with the help of inventory theory.

## 25.1 Inventory Models

In this chapter, we will consider several types of models starting with the deterministic case in the next section. Even though many features of an inventory system involve uncertainty of some kind, it is common to assume much simpler deterministic models for which solutions are found using calculus. Deterministic models also provide a base on which to incorporate assumptions concerning uncertainty. Section 25.3 adds a stochastic dimension to the model with random product demand. Section 25.4 begins discussion of stochastic inventory systems with the single period stochastic model. The model has applications for products for which the ordering process is nonrepeating. The remainder of the chapter addresses models with an infinite time horizon and several assumptions regarding the costs of operation. Sections 25.5 and 25.6 derive optimal solutions for the (*s*, *S*) policy under a variety of conditions. This policy places an order up to level *S* when the inventory level falls to the reorder point *s*. Section 25.7 extends these results to the (*R*, *S*) policy. In this case, the inventory is observed periodically (with a time interval *R*), and is replenished to level *S*.

**Flow, Inventory and Time**

An inventory is represented in the simple diagram of Fig. 1. Items flow into the system, remain for a time and then flow out. Inventories occur whenever the time an individual enters is different than when it leaves. During the intervening interval the item is part of the inventory.


Figure 1. A system component with inventory

For example, say the box in Fig. 1 represents a manufacturing process that takes a fixed amount of time. A product entering the box at one moment leaves the box one hour later. Products arrive at a rate of 100 per hour. Clearly, if we look in the box, we will find some number of items. That number is the inventory level. The relation between flow, time and inventory level that is basic to all systems is

$$\text{Inventory level} = (\text{Flow rate})(\text{Residence time}) \qquad (1)$$

where the flow rate is expressed in the same time units as the residence time. For the example, we have

$$\text{Inventory Level} = (100 \text{ products/hour})(1 \text{ hour}) = 100 \text{ products.}$$

When the factors in Eq. (1) are not constant in time, we typically use their mean values.

Whenever two of the factors in the above expression are given, the third is easily computed. Consider a queueing system for which customers are observed to arrive at an average rate of 10 per hour. When the customer finds the servers busy, he or she must wait. Customers in the system, either waiting or be served, are the inventory for this system. Using a sampling procedure we determine that the average number of customers in the inventory is 5. We ask, how long on the average is each customer in the system? Using the relation between the flow, time and inventory, we determine the answer as 0.5 hours. As we saw in the Chapter 16, Queueing Models, Eq (1) is called Little's Law.

The relation between time and inventory is significant, because very often reducing the throughput time for a system is just as important as reducing the inventory level. Since they are proportional, changing one factor inevitably changes the other.

### The Inventory Level

The inventory level depends on the relative rates of flow in and out of the system. Define $y(t)$ as the rate of input flow at time $t$ and $Y(t)$ the cumulative flow into the system. Define $z(t)$ as the rate of output flow at time $t$ and $Z(t)$ as the cumulative flow out of the system. The inventory level, $I(t)$ is the cumulative input less the cumulative output.

$$I(t) = Y(t) - Z(t) = \int_0^t y(x)dx - \int_0^t z(x)dx \qquad (2)$$

Figure 2 represents the inventory for a system when the rates vary with time.

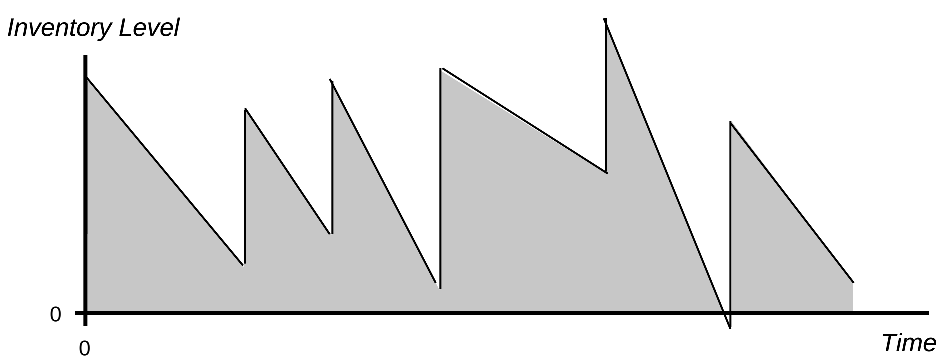


Figure 2. Inventory fluctuations as a function of time

The figure might represent a raw material inventory. The flow out of inventory is a relatively continuous activity where individual items are placed into the production system for processing. To replenish the inventory, an order is placed to a supplier. After some delay time, called the *lead time*, the raw material is delivered in a *lot* of a specified amount. At the moment of delivery, the rate of input is infinite and at other times it is zero. Whenever the instantaneous rates of input and output to a component are not the same, the inventory level changes. When the input rate is higher, inventory grows; when output rate is higher, inventory declines.

Usually the inventory level remains positive. This corresponds to the presence of *on hand inventory*. In cases where the cumulative output exceeds the cumulative input, the inventory level is negative. We call this a *backorder* or *shortage* condition. A backorder is a stored output requirement that is delivered when the inventory finally becomes positive. Backorders may only be possible for some systems. For example, if the item is not immediately available the customer may go elsewhere; alternatively, some items may have an expiration date like an airline seat and can only be backordered up to the day of departure. In cases where backorders are impossible, the inventory level is not allowed to become negative. The demands on the inventory that occur while the inventory level is zero are called *lost sales*.

**Variability, Uncertainty and Complexity**

The are many reasons for variability and uncertainty in inventory systems. The rates of withdrawal from the system may depend on customer demand which is variable in time and uncertain in amount. There may be returns from customers. Lots may be delivered with defects causing uncertainty in quantities delivered. The lead time associated with an order for replenishment depends on the capabilities of the supplier which is usually variable and not known with certainty. The response of a customer to a shortage condition may be uncertain.

Inventory systems are often complex with one component of the system feeding another. Figure 3 shows a simple serial manufacturing system producing a single product.

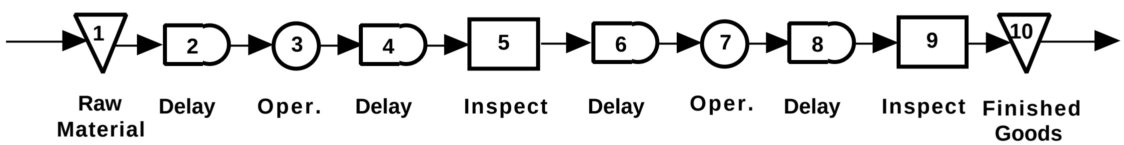


Figure 3. A manufacturing system with several locations for inventories

We identify planned inventories in Fig. 3 as inverted triangles, particularly the raw material and finished goods inventories. Material passing through the production process is often called *work in process* (WIP). These are materials waiting for processing as in the *delay* blocks of the figure, materials undergoing processing in the *operation* blocks, or materials undergoing inspection in the *inspection* blocks. All the components of inventory contribute to the cost of production in terms of handling and investment costs, and all require management attention.

For our analysis, we will often consider one component of the system separate from the remainder, particularly the raw material or finished goods inventories. In reality, rarely can these be managed independently. The material leaving a raw material inventory does not leave the system, rather it flows into the remainder of the production system. Similarly, material entering a finished goods inventory comes from the system. Any analysis that optimizes one inventory independent of the others must provide less than an optimal solution for the system as a whole.

## 25.2 The Deterministic Model

An abstraction to the chaotic behavior of Fig. 2 is to assume that items are withdrawn from the inventory at an even rate $a$, lots are of a fixed size $Q$, and lead time is zero or a constant. The resulting behavior of the inventory is shown in Fig. 4. We use this deterministic model of the system to explain some of the notation associated with inventory. Because of its simplicity, we are able to find an optimal solutions to the deterministic model for several operating assumptions.

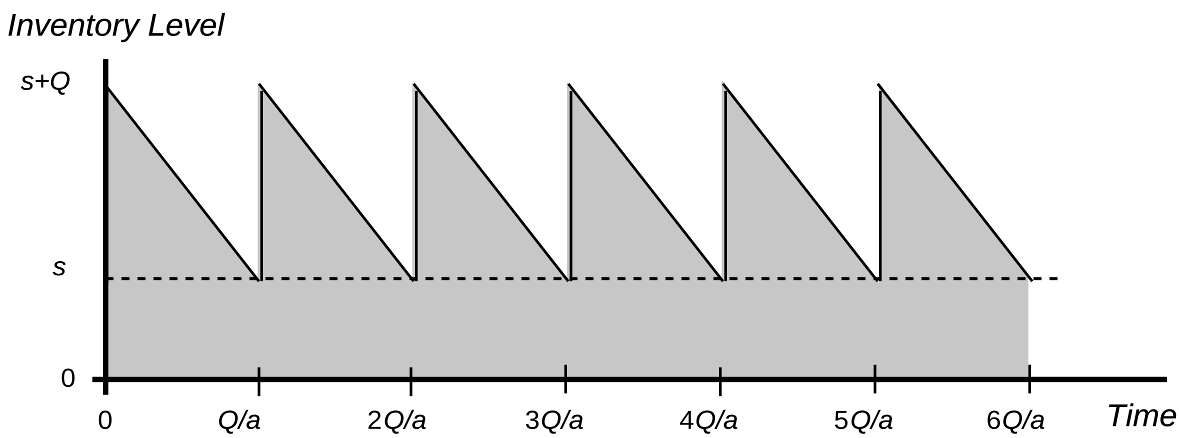


Figure 4. The inventory pattern without uncertainty

*Notation*

This section lists the factors that are important in making decisions related to inventories and establishes some of the notation that is used in this section. Dimensional analysis is sometimes useful for modeling inventory systems, so we provide the dimensions of each factor. Additional model dependent notation is introduced later.

- *Ordering cost* ($c(z)$): This is the cost of placing an order to an outside supplier or releasing a production order to a manufacturing shop. The amount ordered is $z$ and the function $c(z)$ is often nonlinear. The dimension of ordering cost is ($).
- *Setup cost* ($K$): A common assumption is that the ordering cost consists of a fixed cost, that is independent of the amount ordered, and a variable cost that depends on the amount ordered. The fixed cost is called the setup cost and given in ($).
- *Product cost* ($c$): This is the unit cost of purchasing the product as part of an order. If the cost is independent of the amount ordered, the total cost is $cz$, where $c$ is the unit cost and $z$ is the amount ordered. Alternatively, the product cost may be a decreasing function of the amount ordered. ($/unit)

- *Holding cost* ($h$): This is the cost of holding an item in inventory for some given unit of time. It usually includes the lost investment income caused by having the asset tied up in inventory. This is not a real cash flow, but it is an important component of the cost of inventory. If $c$ is the unit cost of the product, this component of the cost is $c\alpha$, where $\alpha$ is the discount or interest rate. The holding cost may also include the cost of storage, insurance, and other factors that are proportional to the amount stored in inventory. ($/unit-time)
- *Shortage cost* ($p$): When a customer seeks the product and finds the inventory empty, the demand can either go unfulfilled or be satisfied later when the product becomes available. The former case is called a lost sale, and the latter is called a backorder. Although lost sales are often important in inventory analysis, they are not considered in this section, so no notation is assigned to it. The total backorder cost is assumed to be proportional to the number of units backordered and the time the customer must wait. The constant of proportionality is $p$, the per unit backorder cost per unit of time. ($/unit-time)
- *Demand rate* ($a$): This is the constant rate at which the product is withdrawn from inventory. (units / time)
- *Lot Size* ($Q$): This is the fixed quantity received at each inventory replenishment. (units)
- *Order level* ($S$): The maximum level reached by the inventory is the order level. When backorders are not allowed, this quantity is the same as $Q$. When backorders are allowed, it is less than $Q$. (units)
- *Cycle time* ($\tau$): The time between consecutive inventory replenishments is the cycle time. For the models of this section $\tau = Q/a$. (time)
- *Cost per time* ($T$): This is the total of all costs related to the inventory system that are affected by the decision under consideration. ($/time)
- *Optimal Quantities* ($Q^*$, $S^*$, $\tau^*$, $T^*$): The quantities defined above that maximize profit or minimize cost for a given model are the optimal solution.

**Lot Size Model with no Shortages**

The assumptions of the model are described in part by Fig. 5, which shows a plot of inventory level as a function of time. The inventory level ranges between 0 and the amount $Q$. The fact that it never goes below 0 indicates that no shortages are allowed. Periodically an order is placed for replenishment of the inventory. The order quantity is $Q$. The arrival of the order is assumed to occur instantaneously, causing the inventory level to shoot from 0 to the amount $Q$. Between orders the inventory decreases at a constant rate $a$. The time between orders is called the cycle time, $\tau$, and is the time required to use up the amount of the order quantity, or $Q/a$.

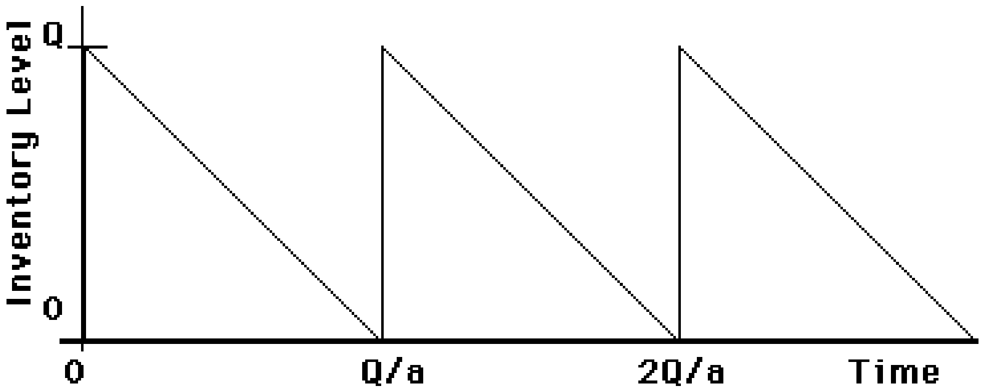


Figure 5. Lot size model with no shortages

The total cost expressed per unit time is

Cost/unit time = Setup cost + Product cost + Holding cost

$$T = \frac{aK}{Q} + ac + \frac{hQ}{2}. \tag{3}$$

In Eq. (3), $\frac{a}{Q}$ is the number of orders per unit time. The factor $\frac{Q}{2}$ is the average inventory level. Setting to zero the derivative of $T$ with respect to $Q$ we obtain

$$\frac{dT}{dQ} = -\frac{aK}{Q^2} + \frac{h}{2} = 0.$$

Solving for the optimal policy,

$$Q^* = \sqrt{\frac{2aK}{h}} \tag{4}$$

$$\text{and } \tau^* = \frac{Q^*}{a} \tag{5}$$

Substituting the optimal lot size into the total cost expression, Eq. (3), and preserving the breakdown between the cost components we see that

$$T^* = \sqrt{\frac{ahK}{2}} + ac + \sqrt{\frac{ahK}{2}} = ac + \sqrt{2ahK} \tag{6}$$

At the optimum, the holding cost is equal to the setup cost. We see that optimal inventory cost is a concave function of product flow through the inventory ($a$), indicating that there is an economy of scale associated with the flow through inventory. For this model, the optimal policy does not depend on the unit product cost. The optimal lot size increases with increasing setup cost and flow rate and decreases with increasing holding cost.

*Example 1*

A product has a constant demand of 100 units per week. The cost to place an order for inventory replenishment is $1000. The holding cost for a unit in inventory is $0.40 per week. No shortages are allowed. Find the optimal lot size and the corresponding cost of maintaining the inventory. The optimal lot size from Eq. (4) is

$$Q^* = \sqrt{\frac{2(100)(1000)}{0.4}} = 707.$$

The total cost of operating the inventory from Eq. (6) is

$$T^* = \$282.84 \text{ per week.}$$

From $Q^*$ and Eq. (5), we compute the cycle time,

$$t^* = 7.07 \text{ weeks.}$$

The unit cost of the product was not given in this problem because it is irrelevant to the determination of the optimal lot size. The product cost is, therefore, not included in $T^*$.

Although these results are easy to apply, a frequent mistake is to use inconsistent time dimensions for the various factors. Demand may be measured in units per week, while holding cost may be measured in dollars per year. The results do not depend on the time dimension that is used; however, it is necessary that demand be translated to an annual basis or holding cost translated to a weekly basis.

## Shortages Backordered

A deterministic model considered in this section allows shortages to be backordered. This situation is illustrated in Fig. 6. In this model the inventory level decreases below the 0 level. This implies that a portion of the demand is backlogged. The maximum inventory level is $S$ and occurs when the order arrives. The maximum backorder level is $Q - S$. A backorder is represented in the figure by a negative inventory level.

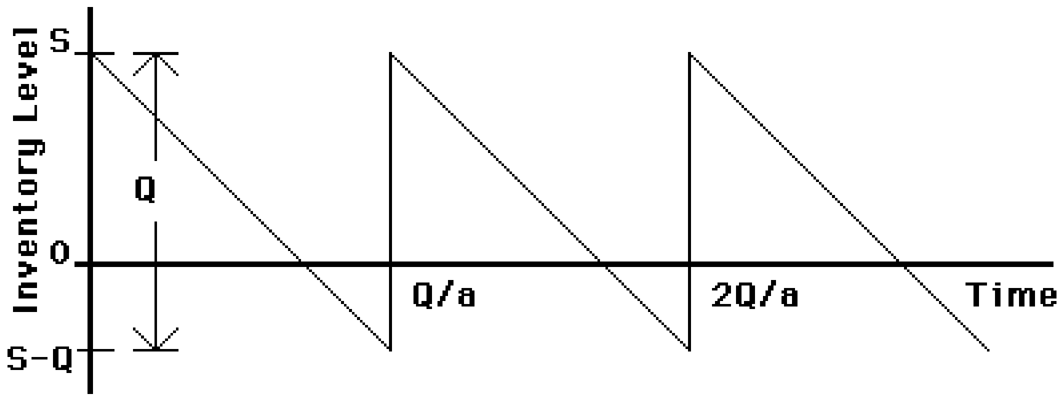


Figure 6 Lot-size model with shortages allowed

The total cost per unit time is

Cost/time = Setup cost + Product cost + Holding cost + Backorder cost

$$T = \frac{aK}{Q} + ac + \frac{hS^2}{2Q} + \frac{p(Q - S)^2}{2Q} \tag{7}$$

The factor multiplying $h$ in this expression is the average on-hand inventory level. This is the positive part of the inventory curve shown in Fig. 6. Because all cycles are the same, the average on-hand inventory computed for the first cycle is the same as for all time. We see the first cycle in Fig. 7.

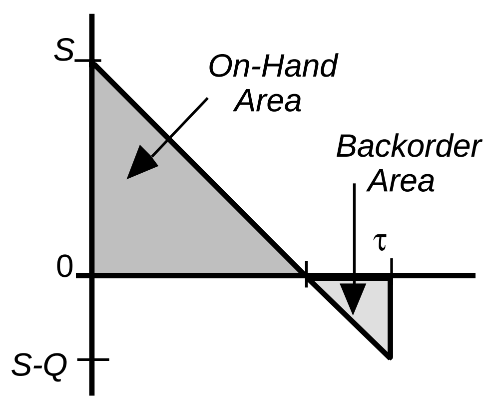


Figure 7. The first cycle of the lot size with backorders model

Defining $O(t)$ as the on-hand inventory level and $\overline{O}$ as the average on-hand inventory

$$\overline{O} = (1/\tau)\int_0^{\tau} O(t)dt = (1/\tau)[\text{On -hand Area}]$$

$$= \left(\frac{a}{Q}\right)\left(\frac{S^2}{2a}\right) = \frac{S^2}{2Q}$$

Similarly the factor multiplying $p$ is the average backorder level, $\overline{B}$, where

$$\overline{B} = (1/\tau)(\text{Backorder Area}) = \frac{(Q-S)^2}{2Q}.$$

Setting to zero the partial derivatives of $T$ with respect to $Q$ and $S$ yields

$$S^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p}{p+h}} \tag{8}$$

$$Q^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p+h}{p}} \tag{9}$$

$$\text{and } \tau^* = \frac{Q^*}{a} \tag{10}$$

Comparing these results to the no shortage case, we see that the optimal lot size and the cycle times are increased by the factor

$$[(p + h)/h]^{1/2}.$$

The ratio between the order level and the lot size depends only on the relative values of holding and backorder cost.

$$S^*/Q^* = \frac{\sqrt{ph}}{p+h} \tag{11}$$

This factor is 1/2 when the two costs are equal, indicating that the inventory is in a shortage position one half of the time.

*Example 2*

We continue Example 1, but now we allow backorders. The backorder cost is $1 per unit-week. The optimal policy for this situation is found with Eqs. (8), (9) and (10).

$$S^* = \sqrt{\frac{2(100)(1000)}{0.4}}\sqrt{\frac{1}{1+0.4}} = 597.61$$

$$Q^* = \sqrt{\frac{2(100)(1000)}{0.4}}\sqrt{\frac{1+0.4}{1}} = 836.66$$

$$t^* = \frac{836.66}{100} = 8.36 \text{ weeks.}$$

Again neglecting the product cost, we find from Eq. (7)

$$T^* = \$239.04 \text{ per week.}$$

The cost of operation has decreased since we have removed the prohibition against backorders. There backorder level is 239 during each cycle.

**Quantity Discounts**

The third deterministic model considered incorporates quantity discount prices that depend on the amount ordered. For this model no shortages are allowed, so the inventory pattern appears as in Fig. 5. The discounts will affect the optimal order quantity. For this model we assume there are $N$ different prices: $c_1, c_2, \ldots, c_N$, with the prices decreasing with the index. The quantity level at which the $k$th price becomes effective is $q_k$, with $q_1$ equal zero. For purposes of analysis define $q_{(N+1)}$ equal to infinity, indicating that the price $c_N$ holds for any amount greater than $q_N$. Since the price decreases as quantity increases the values of $q_k$ increase with the index $k$.

To determine the optimal policy for this model we observe that the optimal order quantity for the no backorder case is not affected by the product price, $c$. The value of $Q_k^*$ would be the same for all price levels if not for the ranges of order size over which the prices are effective. Therefore we compute the optimal lot size $Q^*$ using the parameters of the problem.

$$Q^* = \sqrt{\frac{2aK}{h}}\,. \tag{12}$$

We then find the optimal order quantity for each price range.

Find for each $k$ the value of $Q_k^*$ such that

if $Q^* > q_{k+1}$ then $Q_k^* = q_{k+1}$,

if $Q^* < q_k$ then $Q_k^* = q_k$,

if $q_k \le Q^* < q_{k+1}$ then $Q_k^* = Q^*$

*Optimal Order Quantity* $(Q^{**})$

a. Find the price level for which $Q^*$ lies within the quantity range (the last of the conditions above is true). Let this be level $n^*$. Compute the total cost for this lot size

$$T_{n^*} = \frac{aK}{Q^*} + ac_{n^*} + \frac{hQ^*}{2} . \tag{13}$$

b. For each level $k > n^*$, compute the total cost $T_k$ for the lot size $Q_k^*$.

$$T_k = \frac{aK}{Q_k^*} + ac_k + \frac{hQ_k^*}{2} \tag{14}$$

c. Let $k^*$ be the level that has the smallest value of $T_k$. The optimal lot size $Q^{**}$ is the lot size giving the least total cost as calculated in Steps b and c.

*Example 3*

We return to the situation of Example 1, but now assume quantity discounts. The company from which the inventory is purchased hopes to increase sales by offering a break on the price of the product for larger orders. For an amount purchased from 0 to 500 units, the unit price is $100. For orders at or above 500 but less than 1000, the unit price is $90. This price applies to all units purchased. For orders at or greater than 1000 units, the unit price is $85.

From this data we establish that $N = 3$. Also

$q_1 = 0$ and $c_1 = 100$,

$q_2 = 500$ and $c_2 = 90$,

$q_3 = 1000$ and $c_3 = 85$,

$q_4 = \infty$.

Neglecting the quantity ranges, from Eq. (12) we find the optimal lot size is 707 regardless of price. We observe that this quantity falls in the second price range. All lower ranges are then excluded. We must then compare the cost at $Q = 707$ and $c_2 = 90$, with the cost at $Q = 1000$ and $c_3 = 85$. For the cost $c_2$ we use Eq. (13).

$$T_2 = \$9282 \text{ (for } Q_2^* = 707 \text{ and } c_2 = 90)$$

For the cost $c_3$ we use Eq. (14).

$$T_3 = \$8{,}800 \ \ (\text{for } Q_3^* = 1000 \text{ and } c_3 = 85).$$

Comparing the two costs, we find the optimal policy is to order 1000 for each replenishment. The cycle time associated with this policy is 10 weeks.

**Modeling**

The inventory analyst has three principal tasks: constructing the mathematical model, specifying the values of the model parameters, and finding the optimal solution. This section has presented only the simplest cases, with the model specified as the total cost function. The model can be varied in a number of important aspects. For example, non-instantaneous replenishment rate, multiple products, and constraints on maximum inventory are easily incorporated.

When a deterministic model contains a nonlinear total cost function with only a few variables, the tools of calculus can often be used find the optimal solution. Some assumptions, however, lead to complex optimization problems requiring nonlinear programming or other numerical methods.

The classic lot size formulas derived in this section are based on a number of assumptions that are usually not satisfied in practice. In addition it is often difficult to accurately estimate the parameters used in the formulas. With the admitted difficulties of inaccurate assumptions and parameter estimation, one might question whether the lot size formulas should be used at all. We should point out that whether or not the formulas are used, lot size decisions are frequently required. However abstract the models are, they do recognize important relationships between the various cost factors and the lot size, and they do provide answers to lot sizing questions.

# 25.3 Stochastic Inventory Models

There is no question that uncertainty plays a role in most inventory management situations. The retail merchant wants enough supply to satisfy customer demands, but ordering too much increases holding costs and the risk of losses through obsolescence or spoilage. An order too small increases the risk of lost sales and unsatisfied customers. The water resources manager must set the amount of water stored in a reservoir at a level that balances the risk of flooding and the risk of shortages. The operations manager sets a master production schedule considering the imprecise nature of forecasts of future demands and the uncertain lead time of the manufacturing process. These situations are common, and the answers one gets from a deterministic analysis very often are not satisfactory when uncertainty is present. The decision maker faced with uncertainty does not act in the same way as the one who operates with perfect knowledge of the future.

In this section we deal with inventory models in which the stochastic nature of demand is explicitly recognized. Several models are presented that again are only abstractions of the real world, but whose answers can provide guidance and insight to the inventory manager.

### Probability Distribution for Demand

The one feature of uncertainty considered in this section is the demand for products from the inventory. We assume that demand is unknown, but that the probability distribution of demand is known. Mathematical derivations will determine optimal policies in terms of the distribution.

- *Random Variable for Demand* ($x$): This is a random variable that is the demand for a given period of time. Care must be taken to recognize the period for which the random variable is defined because it differs among the models considered.
- *Discrete Demand Probability Distribution Function* ($P(x)$): When demand is assumed to be a discrete random variable, $P(x)$ gives the probability that the demand equals $x$.
- *Discrete Cumulative Distribution Function* ($F(b)$): The probability that demand is less than or equal to $b$ is $F(b)$ when demand is discrete.

$$F(b) = \sum_{x=0}^{b} P(x)$$

- *Continuous Demand Probability Density Function* ($f(x)$): When demand is assumed to be continuous, $f(x)$ is its density function. The probability that the demand is between $a$ and $b$ is

$$P(a \le X \le b) = \int_a^b f(x)dx.$$

We assume that demand is nonnegative, so $f(x)$ is zero for negative values.

- *Continuous Cumulative Distribution Function* ($F(b)$): The probability that demand is less than or equal to $b$ when demand is continuous.

$$F(b) = \int_0^b f(x)dx$$

- *Standard Normal Distribution Function* ($\phi(x)$ and $\Phi(x)$): These are the density function and cumulative distribution function for the standard normal distribution.
- *Abbreviations*: In the following we abbreviate probability distribution function or probability density function as pdf. We abbreviate the cumulative distribution function as CDF.

*Selecting a Distribution*

An important modeling decision concerns which distribution to use for demand. A common assumption is that individual demand events occur independently. This assumption leads to the Poisson distribution when the expected demand in a time interval is small and the normal distribution when the expected demand is large. Let $a$ be the average demand rate. Then for an interval of time $t$ the expected demand is $at$. The Poisson distribution is then

$$P(x) = \frac{(at)^x e^{-(at)}}{x!}.$$

When $at$ is large the Poisson distribution can be approximated with a normal distribution with mean and standard deviation

$$\mu = at \text{ , and } \sigma = \sqrt{at} \text{ .}$$

Values of $F(b)$ are evaluated using tables for the standard normal distribution. We include these tables at the end of this chapter.

Of course other distributions can be assumed for demand. Common assumptions are the normal distribution with other values of the mean and standard deviation, the uniform distribution, and the exponential distribution. The latter two are useful for their analytical simplicity.

*Finding the Expected Shortage and the Expected Excess*

We are often concerned about the relation of demand during some time period relative to the inventory level at the beginning of the time period. If the demand is less than the initial inventory level, there is inventory remaining at the end of the interval. This is the condition of excess. If the demand is greater than the initial inventory level, we have the condition of shortage.

At some point, assume the inventory level is a positive value $z$. During some interval of time, the demand is a random variable $x$ with pdf, $f(x)$, and CDF, $F(x)$. The mean and standard deviation of this distribution are $\mu$ and $\sigma$, respectively. With the given distribution, we compute the probability of a shortage, $P_S$, and the probability of excess, $P_e$. For a continuous distribution

$$P_S = P\{x > z\} = \int_z^\infty f(x)dx = 1 - F(z) \tag{15}$$

$$P_e = P\{x \le z\} = \int_0^z f(x)dx = F(z) \tag{16}$$

In some cases we may also be interested in the expected shortage, $E_S$. This depends on whether the demand is greater or less than $z$.

$$\text{Items short} = \begin{cases} 0, & \text{if } x \le z \\ x - z, & \text{if } x > z \end{cases}$$

Then $E_S$ is the expected shortage and is

$$E_S = \int_z^\infty (x - z) f(x)dx\,. \tag{17}$$

Similarly for excess, the expected excess is $E_e$

$$E_e = \int_0^z (z - x) f(x)dx$$

The expected excess is expressed in terms of $E_S$

$$E_e = \int_0^\infty (z - x) f(x)dx \; - \; \int_z^\infty (z - x) f(x)dx$$

$$= z - \mu + E_S. \tag{18}$$

For discrete distributions, sums replace the integrals in Eqs. (15) through (18).

$$P_S = P\{x \ge z\} = \sum_{x=z}^{\infty} P(x)dx \; = 1 - F(z), \tag{19}$$

$$P_e = P\{x \le z\} = \sum_{x=0}^{z} P(x)dx \; = F(z). \tag{20}$$

$$E_S = \sum_{x=z}^{\infty} (x - z)P(x)dx. \tag{21}$$

$$E_e = \sum_{x=0}^{z} (z - x)P(x)dx = z - \mu + E_S. \tag{22}$$

*When the Distribution of Demand is Normal*

When the demand during the lead time has a normal distribution, tables are used to find these quantities. Assume the demand during the lead time has a normal distribution with mean $\mu$ and standard deviation $\sigma$. We specify the inventory level in terms of the number of standard deviations away from the mean.

$$z = \mu + k\sigma \text{ or } k = \frac{z - \mu}{\sigma}$$

We have included at the end of this chapter, a table for the standard normal distribution, $\phi(y)$, $\Phi(y)$ and $G(y)$. We have formerly identified the first two of these functions as the pdf and CDF. The third is defined as

$$G(k) = \int_k^{\infty} (y - k)\phi(y)dy = \phi(k) - k[1 - \Phi(k)].$$

Using the relations between the normal distribution and the standard normal, the following relationships hold.

$$f(z) = (1/\sigma)\phi(k) \tag{23}$$

$$F(z) = \Phi(k) \tag{24}$$

$$E_S(z) = \sigma G(k) \tag{25}$$

$$E_e = z - \mu + \sigma G(k) \tag{26}$$

We have occasion to use these results in subsequent examples.

## 25.4 Single Period Stochastic Inventories

This section considers an inventory situation in which the current order for the replenishment of inventory can be evaluated independently of future decisions. Such cases occur when inventory cannot be added later (spares for a space trip, stocks for the Christmas season), or when inventory spoils or becomes obsolete (fresh fruit, current newspapers). The problem may have multiple periods, but the current inventory decision must be independent of future periods. First we assume there is no setup cost for placing a replenishment order, and then we assume that there is a setup cost.

### Single Period Model with No Setup Cost

Consider an inventory situation where the merchant must purchase a quantity of items that is offered for sale during a single interval of time. The items are purchased for a cost $c$ per unit and sold for a price $b$ per unit. If an item remains unsold at the end of the period, it has a salvage value of $a$. If the demand is not satisfied during the interval, there is a cost of $d$ per unit of shortage. The demand during the period is a random variable $x$ with given pdf and CDF. The problem is to determine the number of items to purchase. We call this the *order level*, $S$, because the purchase brings the inventory to level $S$. For this section, there is no cost for placing the order for the items.

The expression for the profit during the interval depends on whether the demand falls above or below $S$. If the demand is less than $S$, revenue is obtained only for the number sold, $x$, while the quantity purchased is $S$. Salvage is obtained for the unsold amount $S - x$. The profit in this case is

$$\text{Profit} = bx - cS + a(S - x) \text{ for } x \le S.$$

If the demand is greater than $S$, revenue is obtained only for the number sold, $S$. A shortage cost of $d$ is expended for each item short, $x - S$. The profit in this case is

$$\text{Profit} = bS - cS - d(x - S) \text{ for } x \ge S.$$

Assuming a continuous distribution and taking the expectation over all values of the random variable, the expected profit is

$$E[\text{Profit}] = b\int_0^S xf(x)dx + b\int_S^\infty Sf(x)dx$$
$$- cS + a\int_0^S (S-x)f(x)dx - d\int_S^\infty (x-S)f(x)dx .$$

Rearranging and simplifying,

$$E[\text{Profit}] = b\mu - cS + a\int_0^S (S-x)f(x)dx - (d+b)\int_S^\infty (x-S)f(x)dx .$$

We recognize in this expression the expected excess, $E_e$, and the expected shortage, $E_s$. The profit is written in these terms as

$$E[\text{Profit}] = b\mu - cS + aE_e - (d+b)E_s \qquad (27)$$

To find the optimal order level, we set the derivative of profit with respect to $S$ equal to zero.

$$\frac{dE[\text{Profit}]}{dS} = -c + a\int_0^S f(x)dx + (d+b)\int_S^\infty f(x)dx = 0.$$

or $$-c + aF(S) + (d+b)[1-F(S)] = 0.$$

The CDF of the optimal order level, $S^*$, is determined by

$$F(S^*) = \frac{b-c+d}{b-a+d} . \qquad (28)$$

This result is sometimes expressed in terms of the purchasing cost, $c$, a holding cost $h$, expended for every unit held at the end of the period, and a cost $p$, expended for every unit of shortage at the end of the period. In these terms the optimal expected cost is

$$E[\text{Cost}] = cS + hE_e + pE_s.$$

The optimal solution has

$$F(S^*) = \frac{p-c}{p+h} . \qquad (29)$$

The two solutions are equivalent if we identify

$h = -a$ = negative of the salvage value

$p = b + d$ = lost revenue per unit + shortage cost.

If the demand during the period has a normal distribution with mean and standard deviation $\mu$ and $\sigma$, the expected profit is easily evaluated for any given order level. The order level is expressed in terms of the number of standard deviations from the mean, or

$$S = \mu + k\sigma.$$

The optimality condition becomes

$$\Phi(k^*) = \frac{b - c + d}{b - a + d} = \frac{p - c}{p + h}. \tag{30}$$

The expected value of profit is evaluated with the expression

$$E[\text{Profit}] = b\mu - cS + a[S - \mu + \sigma G(k)] - (d + b)\sigma G(k). \tag{31}$$

Call the quantity on the right of the Eq. (28) or (29) the *threshold*. Optimality conditions for the order level give values for the CDF. For continuous random variables there is a solution if the threshold is in the range from 0 to 1. No reasonable values of the parameters will result in a threshold less than 0 or larger than 1.

For discrete distributions the optimal value of the order level is the smallest value of $S$ such that

$$E[\text{Profit} \mid S + 1] \leq E[\text{Profit} \mid S + 1].$$

By manipulation of the summation terms that define the expected profit, we can show that the optimal order level is the smallest value of $S$ whose CDF equals or exceeds the threshold. That is

$$F(S^*) \geq \frac{b - c + d}{b - a + d} \text{ or } \frac{p - c}{p + h}. \tag{32}$$

*Example 4: Newsboy Problem*

The classic illustration of this problem involves a newsboy who must purchase a quantity of newspapers for the day's sale. The purchase cost of the papers is $0.10 and they are sold to customers for a price of $0.25. Papers unsold at the end of the day are returned to the publisher for $0.02. The boy does not like to disappoint his customers (who might turn elsewhere for supply), so he estimates a "good will" cost of $0.15 for each customer who is not be satisfied if the supply of papers runs out. The boy has kept a record of sales and shortages, and estimates that the mean demand during the day is 250 and the standard deviation is 50. A Normal distribution is assumed. How many papers should he purchase?

This is a single-period problem because today's newspapers will be obsolete tomorrow. The factors required by the analysis are

$a$ = 0.02, the salvage value of a newspaper,

$b$ = 0.25, the selling price of each paper,

$c$ = 0.10, the purchase cost of each paper,

$d$ = 0.15, the penalty cost for a shortage.

Because the demand distribution is normal, we have from Eq. (30),

$$\Phi(k^*) = \frac{b - c + d}{b - a + d} = \frac{0.25 - 0.10 + 0.15}{0.25 - 0.02 + 0.15} = 0.7895.$$

From the normal distribution table, we find that

$$\Phi(0.80) = 0.7881 \text{ and } \Phi(0.85) = 0.8022.$$

With linear interpolation, we determine $k^* = 0.805$. Then

$$S^* = (0.805)(50) + 250 = 290.2.$$

Rounding up, we suggest that the newsboy should purchase 291 papers for the day. The risk of a shortage during the day is

$$1 - F(S^*) = 0.211.$$

Interpolating in the $G(k)$ column in Table 4, we find that

$$G(k^*) = G(0.805) = 0.1192.$$

Then from Eqs. (25), (26) and (31),

$$E_e = 46.2,\ E_s = 5.96, \text{ and } E[\text{Profit}] = \$32.02 \text{ per day}.$$

*Example 5: Spares Provisioning*

A submarine has a very critical component that has a reliability problem. The submarine is beginning a 1-year cruise, and the supply officer must determine how many spares of the component to stock. Analysis shows that the time between failures of the component is 6 months. A failed component cannot be repaired but must be replaced from the spares stock. Only the component actually in operation may fail; components in the spares stock do not fail. If the stock is exhausted, every additional failure requires an expensive resupply operation with a cost of $75,000 per component. The component has a unit cost of $10,000 if stocked at the beginning of the cruise. Component spares also use up space and other scarce resources. To reflect these factors a cost of $25,000 is added for every component remaining unused at the end of the trip. There is essentially no value to spares remaining at the end of the trip because of technical obsolescence.

This is a single-period problem because the decision is made only for the current trip. Failures occur at random, with an average rate of 2 per year. Thus the expected number of failures during the cruise is 2. The number of failures has a Poisson distribution. The second form of the solution, Eq. (29), is convenient in this case.

$h$ = 25,000, the extra cost of storage.

$c$ = 10,000, the purchase cost of each component.

$p$ = 75,000 the cost of resupply.

Expressed in thousands, the threshold is

$$F(S^*) = \frac{p - c}{p + h} = \frac{75 - 10}{75 + 25} = 0.65.$$

From the cumulative Poisson distribution using a mean of 2, we find

$$F(0) = 0.135,\ F(1) = 0.406,\ F(2) = 0.677,\ F(3) = 0.857.$$

Because this is a discrete distribution, we select the smallest value of $S$ such that the CDF exceeds 0.65. This occurs for $S^* = 2$ which means, somewhat surprisingly, that only two spares should be brought. This is in addition to the component initially installed, so that only on the third failure will a resupply be required. The probability of one or more resupply operations is

$$1 - F(2) = 0.323.$$

The relevance of this model is due in part to the resupply aspect of the problem. If the system simply stopped after the spares were exhausted and a single cost of failure were expended, then the assumption of the linear cost of lost sales would be violated.

## Single Period Model with a Fixed Ordering Cost

When the merchant has an initial source of product and there is a fixed cost for ordering new items, it may be less expensive to purchase no additional items than to order up to some order level. In this section, we assume that initially there are $z$ items in stock. If more items are purchased to increase the stock to a level $S$, a fixed ordering charge $K$ is expended. We want to determine a level $s$, called the *reorder point*, such that if $z$ is greater than $s$ we do not purchase additional items. Such a policy is called the reorder point, order level system, or the ($s$, $S$) system.

We consider first the case where additional product is ordered to bring the inventory to $S$ at the start of the period. The expression for the expected profit is the same as developed previously, except we must subtract the ordering charge and it is only necessary to purchase ($S - z$) items.

$$P_O(z, S) = b\mu - c(S - z) + aE_e[S] - (d + b)E_s[S] - K \qquad (33)$$

We include the argument $S$ with $E_e[S]$ and $E_s[S]$ to indicate that these expected values are computed with the starting inventory level at $S$.

Neither $z$ nor $K$ affect the optimal solution, and as before

$$F(S^*) = \frac{b - c + d}{b - a + d}$$

If no addition items are purchased, the system must suffice with the initial inventory $z$. The expected profit in this case is

$$P_N(z) = b\mu + aE_e[z] - (d + b)E_s[z], \qquad (34)$$

where the expected excess and shortage depend on $z$.

When $z$ equals $S$, $P_N$ is greater than $P_O$ by the amount $K$, and certainly no additional items should be purchased. As $z$ decreases, $P_N$ and $P_O$ become closer. The two expressions are equal when $z$ equals $s$, the optimal reorder point. Then the optimal reorder point is $s*$ where,

$$P_O(s*, S) = P_N(s*)$$

Generally it is difficult to evaluate the integrals that allow this equation to be solved. When the demand has a normal distribution, however, the expected profit in the two cases can be written as a function of the distribution parameters.

Assuming a normal distribution and given the initial supply, $z$, the profit when we replenish the inventory up to the level $S$ is

$$P_O(z, S) = b\mu - c(S - z) + a[S - \mu + \sigma G(k)] - (d + b)[\sigma G(k)] - K \qquad (35)$$

Here $S = \mu + k\sigma$. If we choose not to replenish the inventory, but rather operate with the items on hand the profit is

$$P_N(z) = E[\text{Profit}] = b\mu + a[z - \mu + \sigma G(k_z)] - (d + b)[\sigma G(k_z)]. \quad (31)$$

Here $z = \mu + k_z\sigma$.

We modify the newsboy problem by assuming that the boy gets a free stock of papers each morning. The question is whether he should order more? The cost of placing an order is \$10. In Fig. 8, we have plotted these the costs with and without an order. The profit is low when the initial stock is low and we do not reorder. The two curves cross at about 210. This is the reorder point for the newsboy. If he has 210 papers or less, he should order enough papers to bring his stock to 291. If he has more than 210 papers, he should not restock. The profit for a given day depends on how many papers the boy starts with. The higher of the two curves in Fig. 8 shows the daily profit if one follows the optimal policy. As expected the profit grows with the number of free papers.

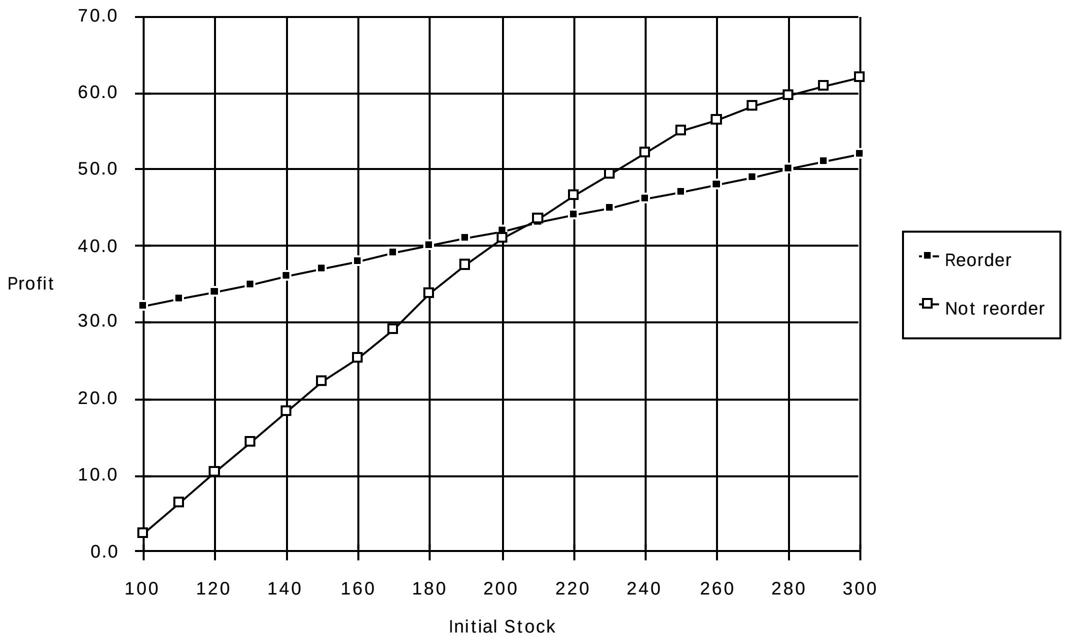


Figure 8. Determining the reorder point for the newsboy problem

*Example 6: Demand with a Uniform Distribution*

The demand for the next period is a random variable with a uniform distribution ranging from 50 to 250 units. The purchase cost of an item is \$100. The selling price is \$150. Items unsold at the end of the period go "on sale" for \$20. All remaining are disposed of at this price. If the inventory is not sufficient, sales are lost, with a penalty equal to the selling price of the item. The current level of inventory is 100 units. Additional items may be ordered at this time; however, a delivery fee will consist of a fixed charge of \$500 plus \$10 per item ordered. Should an order be placed, and if so, how many items should be ordered?

To analyze this problem first determine the parameters of the model.

$c$ = \$110, the purchase cost plus the variable portion of the delivery fee

$K$ = \$500, the fixed portion of the delivery fee

$p$ = \$150, the lost income associated with a lost sale

$h$ = –\$20, the negative of the salvage value of the product.

From Eq. (29), the order level is $S$, such that

$$F(S^*) = \frac{p - c}{p + h} = \frac{150 - 110}{150 - 20} = 0.3077.$$

Setting the CDF for the uniform distribution equal to this value and solving for $S$,

$$F(S) = \frac{S - 50}{250 - 50} = 0.3077 \text{ or } S = 111.5.$$

Rounding up, we select $S^* = 112$.

`Modifying the expected cost function to include the initial stock and the cost of placing and order.

$$C_O = c(S - z) + hE_e[S] + pE_s[S] + K$$

For the uniform distribution ranging from $A$ to $B$,

$$E_e[S] = \frac{1}{(B - A)}\int_A^S (S - x)dx = \frac{(S - A)^2}{2(B - A)}$$

$$E_s[S] = \frac{1}{(B - A)}\int_S^B (x - S)dx = \frac{(B - S)^2}{2(B - A)}$$

$$C_O = c(S - z) + K + \frac{h(S - A)^2 + p(B - S)^2}{2(B - A)}$$

When no order is placed, the purchase cost and the reorder cost terms drop out and $z$ replaces $S$.

$$C_N = \frac{h(z - A)^2 + p(B - z)^2}{2(B - A)}.$$

Evaluating $C_O$ with the order level equal to 112, we find that

$$C_O = 19{,}729 - 110z.$$

Expressing $C_N$ entirely in terms of $z$,

$$C_N = -0.05(z - 50)^2 + 0.375(250 - z)^2$$

Setting $C_O$ equal to $C_N$, substituting $s$ for $z$, we solve for the optimal reorder point.

$$19729 - 110s = -0.05(s - 50)^2 + 0.375(250 - s)^2$$

$$0.325s^2 - 72.5s + 3543.3 = 0$$

Solving the quadratic[1] we find the solutions

$$s = 150.8 \text{ and } s = 72.3.$$

The solution lying above the order level is meaningless, so we select the reorder point of 72. At this point, for

$$s = 72.3, \text{ we have } C_N = C_O = 11{,}814.$$

Because the current inventory level of 100 falls above the reorder point, no additional inventory should be purchased. If there were no fixed charge for delivery, the order would be for 12 units.

*Example 7: Demand with an Exponential Distribution*

Consider the situation of Example 6 except that demand has an exponential distribution with a mean value $\mu = 150$. At the optimal order level

$$F(S^*) = 1 - exp(-S/\mu) = 0.3077.$$

Solving for $S$, we get

$$S = -\mu[ln(1 - 0.3077)] = 55.17.$$

The difference between $s$ and $S$ for the exponential distribution is approximately

$$\Delta = S - s = \sqrt{\frac{2\mu K}{c + h}} = \sqrt{\frac{2(150)(500)}{100 - 20}} = 41$$

$$s = 56 - 41 = 15$$

For this distribution of demand, the current inventory of 100 is considerably above both the reorder point and the order level. Certainly an order should not be placed.

[1] The solution to the quadratic $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac]}}{2a}$.

# 25.5 The (*s*, *Q*) Inventory Policy

We now consider inventory systems similar to the deterministic models presented in Section 25.2, but allow the demand to be stochastic. There are a number of ways one might operate an inventory system with random demand. At this time, we consider the (*s*, *Q*) inventory policy, alternatively called the *reorder point, order quantity* system. Figure 9 shows the inventory pattern determined by the (*s*, *Q*) inventory policy. The model assumes that the inventory level is observed at all times. This is called *continuous review*. When the level declines to some specified *reorder point*, *s*, an order is placed for a *lot size*, *Q*. The order arrives to replenish the inventory after a *lead time*, *L*.

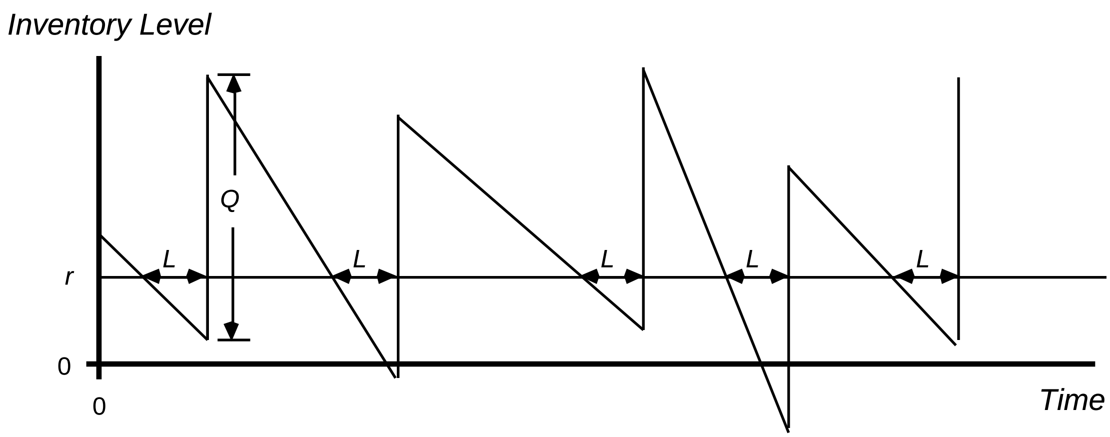


Figure 9. Inventory Operated with the reorder point-lot size Policy

**Model**

The values of *s* and *Q* are the two decisions required to implement the policy. The lead time is assumed known and constant. The only uncertainty is associated with demand. In Fig. 9, we show the decrease in inventory level between replenishments as a straight line, but in reality the inventory decreases in a stepwise and uneven fashion due to the discrete and random nature of the demand process.

If we assume that *L* is relatively small compared to the expected time required to exhaust the quantity *Q*, it is likely that only one order is outstanding at any one time. This is the case illustrated in the figure. We call the period between sequential order arrivals an order cycle. The cycle begins with the receipt of the lot, it progresses as demand depletes the inventory to the level *s*, and then it continues for the time *L* when the next lot is received. As we see in the figure, the inventory level increases instantaneously by the amount *Q* with the receipt of an order.

In the following analysis, we are most concerned with the possibility of shortage during an order cycle, that is the event of the inventory level falling below zero. This is also called the stockout event. We assume shortages are backordered and are satisfied when the next replenishment arrives. To determine probabilities of shortages, one need only be concerned about the random variable that is the demand during the lead time interval. This is the random variable $X$ with pdf, $f(x)$, and CDF $F(x)$. The mean and standard deviation of the distribution are $\mu$ and $\sigma$ respectively. The random demand during the lead time gives rise to the possibility that the inventory level will be depleted before the replenishment arrives. With the *average* rate of demand equal to $a$, the mean demand during the lead time is

$$\mu = aL$$

A shortage will occur if the demand during the period $L$ is greater that $s$. This probability, defined as $P_S$, is

$$P_S = P\{x > s\} = \int_s^\infty f(x)dx = 1 - F(s).$$

The service level is the probability that the inventory will not be depleted during one order cycle, or

$$\text{Service level} = 1 - P_S = F(s).$$

In practical instances the reorder point is significantly greater than the mean demand during the lead time so that $P_S$ is quite small. The safety stock, $SS$, is defined as

$$SS = s - \mu.$$

This is the inventory maintained to protect the system against the variability of demand. It is the *expected* inventory level at the end of an order cycle (just before a replenishment arrives). This is seen in Fig. 10, where we show the $(s, Q)$ policy for deterministic demand. This figure will also be useful for the cost analysis of the system.

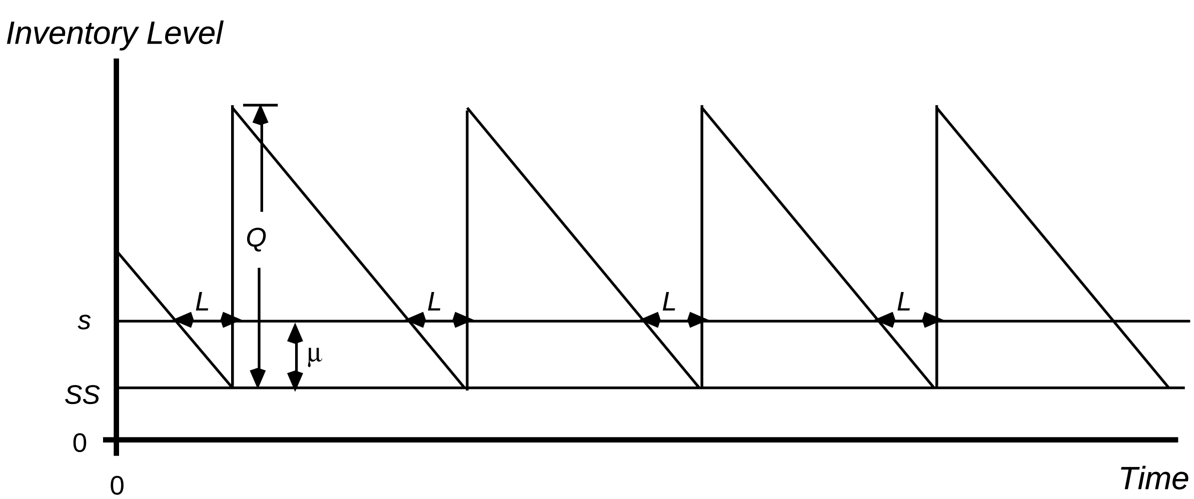


Figure 10. The $(s, Q)$ policy for deterministic demand

## General Solution for the (*s*, *Q*) Policy

We develop here a general cost model for the (*s*, *Q*) policy. The model and its optimal solution depends on the assumption we make regarding the cost effects of shortage. The model is approximate in that we do not explicitly model all the effects of randomness. The principal assumption is that stockouts are rare, a practical assumption in many instances. In the model we use the same notation as for the deterministic models of Section 25.2. Since demand is a random variable, we use *a* as the time averaged demand rate per unit time.

When we assume that the event of a stockout is rare and inventory declines in a continuous manner between replenishments, the average inventory is approximately

$$\text{Average inventory level} = \frac{Q}{2} + s - \mu.$$

Because the per unit holding cost is *h*, the holding cost per unit time is

$$\text{Expected holding cost per unit time} = h\left(\frac{Q}{2} + s - \mu\right).$$

With the backorder assumption, the time between orders is random with a mean value of *Q/a*. The cost for replenishment is *K*, so the expected replenishment cost per unit time is

$$\text{Expected replenishment cost per unit time} = \frac{Ka}{Q}.$$

With the (*s*, *Q*) policy and the assumption that *L* is relatively smaller than the time between orders, *Q/a*, the shortage cost per cycle depends only on the reorder point. We call this $C_S$, and we observe that it is a function of the reorder point *s*. We investigate several alternatives for the definition of this shortage cost. Dividing this cost by the length of a cycle we obtain

$$\text{Expected Shortage cost per unit time} = \frac{a}{Q} C_S.$$

Combining these terms we have the general model for the expected cost of the (*s*, *Q*) policy.

$$\text{EC}(s, Q) = h\left(\frac{Q}{2} + s - \mu\right) \quad \text{Inventory cost}$$

$$+ \frac{Ka}{Q} \quad \text{Replenishment cost}$$

$$+\ \frac{a}{Q} C_s \qquad \text{Shortage cost} \qquad (37)$$

There are two variables in this cost function, *Q* and *s*. To find the optimal policy that minimizes cost, we take the partial derivatives of the expected cost, Eq. (37), with respect to each variable and set them equal to zero. First, the partial derivative with respect to *Q* is

$$\frac{\partial EC}{\partial Q} = \frac{h}{2} - \frac{a(K + C_s)}{Q^2} = 0$$

or $$Q^* = \sqrt{\frac{2a(K + C_s)}{h}} \qquad (38)$$

We have a general expression for the optimal lot size that depends on the cost due to shortages. Taking the partial derivative with respect to the variable *s*,

$$\frac{\partial EC}{\partial s} = h + \frac{a}{Q}\left(\frac{\partial C_s}{\partial s}\right) = 0,$$

or $$\frac{\partial C_s}{\partial s} = -\frac{hQ}{a} \qquad (39)$$

The solution for the optimal reorder point depends on the functional form of the cost of shortage. We consider four different cases in the remainder of this section[2].

**Case of a Fixed Cost per Stockout**

In this case, there is a cost $\pi_1$ expended whenever there is the event of a stockout. This cost is independent of the number of items short, just on the fact that a stockout has occurred. The expected cost per cycle is

$$C_s = \pi_1 P\{x > s\} = \pi_1 \left( \int_s^\infty f(x)dx \right). \qquad (40)$$

Now the partial derivative of Eq. (40) with respect to *s* is

$$\frac{\partial C_s}{\partial s} = -\pi_1 f(s).$$

Combining Eq. (39) with Eq. (40), we have for the optimal value of *s*

$$\frac{\partial C_s}{\partial s} = -\pi_1 f(s^*) = -\frac{hQ}{a},$$

[2]In this article we follow the development in Peterson and Silver [1979], Chapter 7.

$$\text{or} \quad f(s^*) = \frac{hQ}{\pi_1 a}, \tag{41}$$

$$\text{and} \quad C_s = \pi_1(1 - F(s^*)). \tag{42}$$

Equation (41) is a condition on the value of the pdf at the optimal reorder point. If no values of the pdf satisfy this equality, select some minimum safety level as prescribed by management. The pdf may satisfy this condition at two different values. It can be shown that the cost function is minimized when $f(x)$ is decreasing, so for a unimodal pdf, select the greater of the two solutions.

Equation (41) specifying the optimal $s^*$ together with the Eq. (38) for $Q^*$ define the optimal control parameters. If one of the parameters are given at a perhaps not optimal value, these equations yield the optimum for the other parameter. If both parameters are flexible, a successive approximation method, as illustrated in Example 13, is used to find values of $Q$ and $s$ that solve the problem.

*Example 8: Optimal reorder point given the order quantity ($\pi_1$ Given)*

The monthly demand for a product has a normal distribution with a mean of 100 and a standard deviation of 20. We adopt a continuous review policy in which the order quantity is the average demand for one month. The interest rate used for time value of money calculations is 12% per year. The purchase cost of the product is $1000. When it is necessary to backorder, the cost of paperwork is estimated to be $200, independent of the number backordered. Holding cost is estimated using the interest cost of the money invested in a unit of inventory. The lead time for this situation is 1 week. The fixed order cost is $800. Find the optimal inventory policy.

We must first adopt a time dimension for those data items related to time. Here we use 1 month. For this selection,

$a$ = 100 units/month

$h$ = 1000(0.01) = $10/unit-month, the unit cost multiplied by the interest rate (interest rate is 12%/12 = 1% per month)

$\pi_1$ = $1000, the backorder cost, which is independent in time and number

$K$ = $800, the order cost.

We must also describe the distribution of demand during the lead time. For convenience we assume that 1 month has 4 weeks and that the demands in the weeks are independent and identically distributed normal variates. With these assumptions the weekly demand has

$$\mu = 100/4 = 25, \text{ and } \sigma^2 = 20^2/4 = 100 \text{ or } \sigma = 10.$$

The problem specifies the value of $Q$ as 1 month's demand; thus $Q = 100$. Using this value in Eq. (41), we find the associated optimal reorder point.

or $$f(s^*) = \frac{hQ}{\pi_1 a} = \frac{(10)(100)}{(1000)(100)} = 0.01.$$

The pdf of the standard normal distribution is related to a general normal distribution as

$$f(s) = (1/\sigma)\phi(k) \text{ or } \phi(k) = \sigma f(s)$$

Then in terms of the standard normal we have

$$\phi(k^*) = \sigma \frac{hQ}{\pi_1 a} = (10)(0.01) = 0.1.$$

We look this up in the standard normal table provided at the end of this chapter to discover $k^* = \pm 1.66$. Taking the larger of the two possibilities we find

$$s^* = \mu + (1.66)\sigma = 25 + 1.66\,(10) = 41.6$$

or 42 (conservatively rounded up). This is the optimal reorder point for the given value of $Q$.

**Case of a Charge per Unit Short**

In some cases, we may also be interested in the expected number of items backordered during an order cycle, $E_S$. This depends on the demand during the lead time.

$$\text{Items backordered} = \begin{cases} 0, & \text{if } x \le s \\ x - s, & \text{if } x > s \end{cases}$$

Therefore, the expected shortage is

$$E_S = \int_s^\infty (x - s) f(x)dx.$$

For this situation, we assume a cost $\pi_2$ is expended for every unit short in a stockout event. The expected cost per cycle is

$$C_S = \pi_2 E_S.$$

Now the partial derivative with respect to $s$ is

$$\frac{\partial C_s}{\partial s} = -\pi_2 \left( \int_s f(x)dx_0 \right) = -\pi_2(1 - F(s)).$$

From Eq. (41), the optimal value of $s$ must satisfy

$$\frac{\partial C_s}{\partial s} = -\pi_2(1 - F(s^*)) = -\frac{hQ}{a}$$

or

$$F(s^*) = 1 - \frac{hQ}{\pi_2 a}. \quad (43)$$

In this case, we have a condition on the CDF at the optimal reorder point. If the expression on the right is less than zero, use some minimum reorder point specified by management.

For a given value of $s$, the optimal order quantity is determined from Eq. (38) by substituting the value of $C_s$.

$$C_s = \pi_2 E_s = \pi_2 \left( \int_s^\infty (x - s^*) f(x) dx \right). \quad (44)$$

This integral is difficult to compute except for simple distributions. It is evaluated with tables for the normal random variable using Eq. (25).

Managers may find it difficult to specify the shortage cost $\pi_2$. It is easier to specify that the inventory meet some service level. One might require that the inventory meet demands from stock in 99% of the inventory cycles. The service level is actually the value of $F(s)$. Given values of $h$, $Q$ and $a$, one can compute with Eq. (43) the implied shortage cost for the given service level.

*Example 9: Optimal reorder point given the order quantity ($\pi_2$ Given)*

We consider again Example 8, but change the cost structure for backorders. Now we assume that we must treat each backordered customer separately. The cost of paperwork and good will is estimated to be $200 per unit backordered. This is $\pi_2$. The optimal policy is governed by Eq. (43).

$$F(s^*) = 1 - \frac{hQ}{\pi_2 a} = 1 - \frac{(10)(100)}{(200)(100)} = 0.95.$$

We know that the probabilities for a normal distribution is related to the standard normal distribution by

$$F(s) = \phi\left(\frac{s - \mu}{\sigma}\right).$$

$$\phi(k^*) = 0.95.$$

From the normal distribution table we find that this is associated with a standard normal variate of $z = 1.64$. The reorder point is then

$$s^* = \mu + (1.64)\sigma = 25 + 1.64\,(10) = 41.4$$

or 42 (conservatively rounded up). This is the optimal for the given value of $Q$.

**Case of a Charge per Unit Short per Unit Time**

When the backorder cost depends not only on the number of backorders but the time a backorder must wait for delivery, we would like to compute the expected unit-time of backorders for an inventory cycle. When the number of backorders is $x - s$ and the average demand rate is $a$, the average time a customer must wait for delivery is

$$\frac{x-s}{2a}.$$

The resulting unit-time measure for backorders is

$$\frac{(x-s)^2}{2a}.$$

Integrating we find the expected value $T_S$, where

$$T_S = \frac{1}{2a}\int_s^\infty (x-s)^2 f(x)dx. \tag{45}$$

We consider here the case when a cost $\pi_3$ is expended for every unit short per unit of time. The expected cost per cycle is

$$C_S = \pi_3 T_S. \tag{46}$$

Now the partial derivative of $C_S$ with respect to $s$ is

$$\frac{\partial C_s}{\partial s} = -\frac{\pi_3}{a}\left(\int_s^\infty (x-s)f(x)dx\right) = -\frac{\pi_3 E_s}{a}.$$

From Eq. (41), the optimal value of $s$ must satisfy

$$\frac{\partial C_s}{\partial s} = -\frac{\pi_3 E_s}{a} = -\frac{hQ}{a}$$

or

$$E_S(s^*) = \frac{hQ}{\pi_3}. \tag{47}$$

We have added the parameter $s^*$ to the expected shortage to indicate its value is a function of the reorder point. Note that Silver et al. [1998] report the result $E_S(s^*) = \dfrac{Qh}{h+\pi_3}$ which is derived using a more accurate representation of the average inventory. The two results are approximately the same when $\pi_3 >> h$, as assumed here.

*Example 10: Optimal reorder point given the order quantity ($\pi_3$ Given)*

We consider again Example 8, but now we assume that \$1000 is expended per unit backorder per month. This is $\pi_3$. The optimal policy is governed by Eq. (47).

$$E_S(s^*) = \frac{hQ}{\pi_3} = \frac{(10)(100)}{1000} = 1.$$

When the demand is governed by the normal distribution, the expected shortage at the optimum is

$$E_S(s^*) = \sigma G(k^*) = 1$$

$$\text{where } k^* = \frac{s^* - \mu}{\sigma}$$

$$\text{or } G(k^*) = 0.1$$

From the table at the end of the chapter

$$k^* = 0.9.$$

The reorder point is then

$$s^* = \mu + (0.9)\sigma = 25 + 9 = 34$$

This is the optimum for the given value of $Q$.

**Lost Sales Case**

In this case sales are not backordered. A customer that arrives when there is no inventory on hand leaves without satisfaction, and the sale is lost. When stock is exhausted during the lead time, the inventory level rises to the level $Q$ when it is finally replenished. The effect of this situation is to raise the average inventory level by the expected number of shortages in a cycle, $E_S$. We also experience a shortage cost based on the number of shortages in a stockout event. We use $\pi_L$ to indicate the cost for each lost sale. For the case of lost sales the approximate expected cost is

$$\text{EC}(Q, s) = h\left(\frac{Q}{2} + s - \mu + E_s\right) \quad \text{Inventory cost}$$

$$+ \frac{aK}{Q} \quad \text{Replenishment cost}$$

$$+\ \frac{a\pi_L}{Q} E_s \qquad \text{Shortage cost} \qquad (48)$$

Here we are neglecting the fact that with lost sales, not all the demand is met. The number of orders per unit time is slightly less than $a/Q$. Taking partial derivatives with respect to $Q$ and $s$ we find the optimal lot size is

$$Q^* = \sqrt{\frac{2a(K + \pi_L E_s)}{h}} \qquad (49)$$

$$\frac{\partial EC}{\partial s} = h\left(1 + \frac{\partial E_s}{\partial s}\right) + \frac{a\pi_L}{Q}\left(\frac{\partial E_s}{\partial s}\right) = 0,$$

or

$$\frac{\partial E_s}{\partial s} = -\frac{hQ}{hQ + \pi_L a}$$

or

$$(1 - F(s^*)) = \frac{hQ}{hQ + \pi_L a}$$

$$F(s^*) = 1 - \frac{hQ}{hQ + \pi_L a} = \frac{\pi_L a}{hQ + \pi_L a} \qquad (50)$$

*Example 11: Optimal reorder point given the order quantity ($\pi_L$ Given)*

We consider Example 8 again, but now we assume that the sale is lost given a stockout. We charge $2000 for every lost sale. This is $\pi_L$. The optimal policy is governed by Eq. (50).

$$F(s^*) = \frac{\pi_L a}{hQ + \pi_L a} = \frac{(2000)(100)}{(10)(100) + (2000)(100)} = 0.995$$

From the table at the end of the chapter

$$k^* = 2.58.$$

The reorder point is then

$$s^* = \mu + (2.58)\sigma = 25 + 22.6 = 47.6$$

This is the optimum for the given value of $Q$.

**Summary**

We have found in this section solutions for several assumptions regarding the costs due to shortages. These are summarized below for easy use. The optimal reorder point requires one to find the value $s^*$ that corresponds to

$f(s^*)$, $F(s^*)$ or $E_S(s^*)$ equaling some simple function of the problem parameters.

The optimal order quantity for each case depends on the shortage cost, $C_S$, and is given by

$$Q^* = \sqrt{\frac{2a(K + C_s)}{h}}$$

This equation is used directly when a value of $s$ is specified. It is used iteratively when the optimum for both $s$ and $Q$ is required.

Table 1. The ($s$, $Q$) Policy for Continuous Distributions

| Situation | $C_S$ | Optimal reorder point | Normal solution |
|---|---|---|---|
| Fixed cost per stockout ($\pi_1$) | $\pi_1(1 - F(s))$ | $f(s^*) = \frac{hQ}{\pi_1 a}$ | $\phi(k^*) = \frac{\sigma hQ}{\pi_1 a}$ |
| Charge per unit Short ($\pi_2$) | $\pi_2 E_S$ | $F(s^*) = 1 - \frac{hQ}{\pi_2 a}$ | $\phi(k^*) = 1 - \frac{hQ}{\pi_2 a}$ |
| Charge per unit short per unit time ($\pi_3$) | $\pi_3 T_S$ | $E_S(s^*) = \frac{hQ}{\pi_3}$ | $G(k^*) = \frac{hQ}{\sigma\pi_3}$ |
| Charge per unit of lost sales ($\pi_L$) | $\pi_L E_S$ | $F(s^*) = \frac{\pi_L a}{hQ + \pi_L a}$ | $\phi(k^*) = \frac{\pi_L a}{hQ + \pi_L a}$ |

**Determination of the Order Quantity**

Up until this point, all our examples have determined the reorder point given the order quantity. The following examples illustrate the determination of the order quantity when the reorder point is given, and the determination of optimal values for both variables simultaneously.

*Example 12: Optimal order quantity given the reorder point*

We continue from Example 9 in which the shortage cost is $\pi_2$ = \$200 per unit short. The demand during the lead time is normal with $\mu$ = 25 and $\sigma$ = 10. If the reorder point is fixed at 50, what is the optimal order quantity?

For a normal distribution the expected shortage cost is

$$C_S = \pi_2 \sigma G(k_S)$$

where $k_S = (s - \mu)/\sigma$. For $s = 50$ and $k_S = 2.5$, $G(2.5) = 0.0020$, $E_S = 0.020$, $C_S = 4$. Then the optimal order quantity is

$$Q^* = \sqrt{\frac{2a(K + C_s)}{h}} = \sqrt{\frac{2(100)(800 + 4)}{10}} = 126.8$$

or 127 (conservatively rounded up).

*Example 13: Both optimal order quantity and reorder point*

In the previous examples we fixed one of the decisions and found the optimal value of the other. We need an iterative procedure to find both, $Q^*$ and $s^*$. We use the expression below sequentially.

$$Q = \sqrt{\frac{2a(K + C_s)}{h}}, \quad \phi(k_S) = 1 - \frac{hQ}{\pi_2 a}, \quad C_S = \pi_2 \sigma G(k_S)$$

The first step is to assume $C_S = 0$ and to find the corresponding optimal order quantity.

$$Q^1 = 126.5.$$

Using this value of $Q^1$, we find the optimal reorder point

$$k_S = 1.53 \text{ or } s = 40.3$$

The expected shortage per period with this reorder point is

$$C_S = \pi_2 \sigma G(1.53) = (200)(10)(0.02736) = 54.72$$

For this value of $C_S$, we have

$$Q^2 = 130.7.$$

Using this value of $Q^2$, we find the optimal reorder point

$$k_S = 1.51 \text{ or } s = 40.1$$

Computing the associated $C_S$ we find

$$Q^3 = 130.9.$$

It appears that the values are converging, so we adopt the policy

$$Q^* = 131 \text{ and } s^* = 40.$$

# 25.6 Variations on the (*s*, *Q*) Model

### Reorder Point Based on Inventory Position

In the foregoing, we have assumed that a replenishment order is to be placed whenever the inventory level reaches the reorder point. A move practical idea is to use the *inventory position* rather than the inventory level as an indicator. The inventory position is the inventory level plus the quantity on order. The difference is illustrated in Fig. 11. We note that inventory level is the same as inventory position when there are no outstanding orders. In the early cycles of the figure, the inventory level crosses the reorder point at the same time as the inventory position, and the same order pattern is obtained using either measure. Basing the order on the inventory level fails, however, when there is a lead time demand larger than the order quantity, as in the last cycle of the figure. In this case the inventory level falls below the reorder point and never reaches it again. Using the inventory position, however, allows two orders to be placed in quick succession, thus keeping the inventory in control.

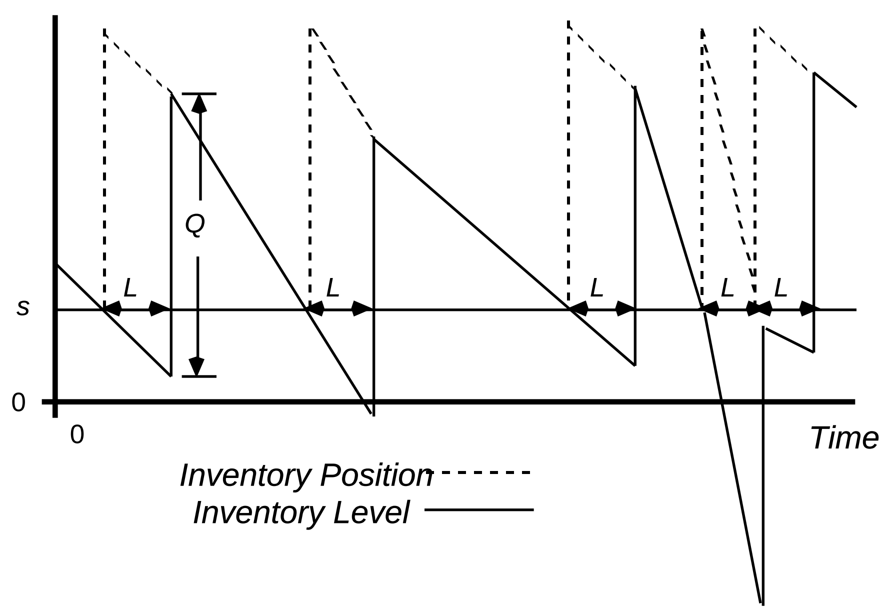


Figure 11. Using inventory position as a measure for placing orders

Using the inventory position in this manner, also allows us to drop the requirement that the lot size be very much greater than the average demand during the lead time. The results in the table can be used even in cases where the lot size is small in relation to the lead time demand. The primary assumption for the derivations is that the probability of a stockout be small. This probability depends on the reorder point and not the lot size.

When the lot size is small, there may be many outstanding orders at any given time, emphasizing the need to track the inventory position. A particularly interesting case is when the lot size is 1. This implies that a replenishment order is placed whenever an item is withdrawn from inventory.

**Discrete Demand During the Lead Time**

The results of the table were derived for continuous distributions. In fact the items in an inventory are usually discrete, and a discrete demand distribution may be more appropriate. This is particularly true when the reorder point is relatively small. For the discrete distribution $p(x)$ is the probability that the random demand during the lead time takes the value $x$. $F(x)$ is the probability that the demand is less than or equal to $x$. The expected shortage and expected unit-time shortage are

$$E_S = \sum_{x=s+1}^{\infty} (x-s)\,p(x) \tag{51}$$

$$T_S = \frac{1}{2a}\sum_{x=s+1}^{\infty} (x-s)^2\,p(x) \tag{52}$$

Table 2. The ($s$, $Q$) Policy for Discrete Distributions

| Situation | $C_S$ | Optimal reorder point |
|---|---|---|
| Fixed cost per stockout ($\pi_1$) | $\pi_1(1 - F(s))$ | $p(s^* - 1) > \dfrac{hQ}{\pi_1 a} \geq p(s^*)$ |
| Charge per unit short ($\pi_2$) | $\pi_2 E_S$ | $F(s^*) \leq 1 - \dfrac{hQ}{\pi_2 a} < F(s^*+1)$ |
| Charge per unit short per unit time ($\pi_3$) | $\pi_3 T_S$ | $E_S(s^*-1) > \dfrac{hQ}{\pi_3} \geq E_S(s^*)$ |
| Charge per unit of lost sales ($\pi_L$) | $\pi_L E_S$ | $F(s^*) \leq 1 - \dfrac{hQ}{hQ + \pi_L a} < F(s^*+1)$ |

**Lead Time a Random Variable**

Previously we assumed that lead time is a constant. Indeed this is a very desirable characteristic of an inventory system. The lead time may actually be uncertain in duration due to variability in shipping times, material availability and supplier processing times.

Let lead time be a random variable $Y$ with pdf $h(y)$, and let demand be the random variable $X$ with joint pdf $g(x, y)$ -- that is, the demand distribution depends on the lead time. The marginal distribution of demand during the lead time is

$$f(x) = \int_0^\infty g(x,y)h(y)dy \tag{53}$$

This pdf can then be used in conjunction with the standard normal distribution table to determine approximate solutions.

## 25.7 The (*R*, *S*) Inventory Policy

A different way to manage a stochastic inventory system is illustrated in Fig. 12 This is called a periodic review policy in that the inventory level is only observed at intervals of *R*. If the inventory is at level *y*, a quantity $S - y$ is ordered to bring the inventory position to ***S*.** *S* is called the *order level*. After a lead time interval *L*, the replenishment order is delivered. Figure 12 shows the inventory position with dotted lines and the inventory level with solid lines.

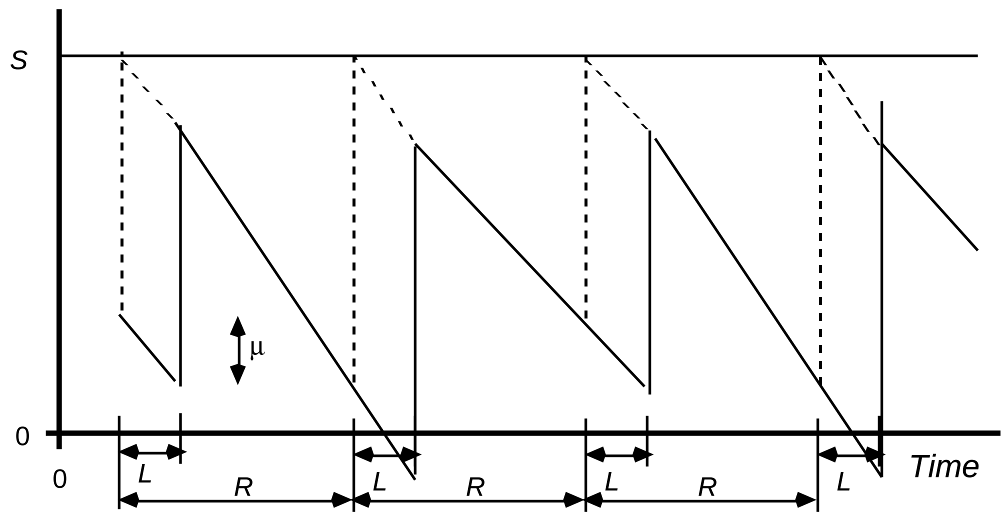


Figure 12. The (*R*, *S*) inventory policy

The analysis of this policy is much like that for the (*s*, *Q*) policy. For the (*s*, *Q*) policy, the reorder point *s* is set to protect against the possibility of shortage during the lead time *L*. For the (*R*, *S*) policy, the order level *S* is set to protect against a shortage in the time interval $R + L$. In the event of a particular order at time *t*, the lowest inventory that is affected by that order occurs at time $t + R + L$. The quantity *S* must be large enough to keep the probability of a shortage in that time interval small. The (*R*, *S*) policy is much more affected by variability than the (*s*, *Q*) policy because of the longer interval. The advantage of the policy is that it does not require continuous review.

To analyze this system, we define the demand in the interval $R + L$ to be the random variable *X*. The pdf and CDF are $f_P(x)$ and $F_P(x)$, respectively. The mean and variance during the interval are $\mu_P$ and $\sigma_P$ (the subscript "P" stands for periodic). The cost (per unit time) of operation of the inventory system expressed in terms of *R* and *S* is

$$EC(R, S) = h\left(\frac{aR}{2} + S - \mu\right) \quad \text{Inventory cost}$$

$$+\ \frac{K}{R} \qquad \text{Replenishment cost}$$

$$+\ \frac{1}{R}C_S \qquad \text{Shortage cost} \qquad (54)$$

Evaluation of the shortage cost depends on the assumption of the costs experienced in the event of a stockout.

Fixed cost per stockout: $$C_s = \pi_1(1 - F(S)) = \pi_1 \left( \int_S^{\infty} f_P(x)dx \right) \qquad (55)$$

Charge per unit short: $$C_s = \pi_2 E_s = \pi_2 \left( \int_S^{\infty} (x - s) f_P(x)dx \right) \qquad (56)$$

Charge per unit short per unit time:

$$C_s = \pi_3 T_s = \left( \frac{1}{2a} \int_S^{\infty} (x - s)^2 f_P(x)dx \right) \qquad (57)$$

Charge per lost sale: $$C_s = \pi_L E_s = \pi_L \left( \int_S^{\infty} (x - s) f_P(x)dx \right) \qquad (58)$$

We cannot determine the optimal *R* easily in this case because $C_s$ depends on *R*. In particular the distribution of demand in the expressions for $C_s$ have a mean and variance that depends on the interval $L + R$. If we assume that the review interval *R* is set elsewhere, the determination of the optimal order level follows exactly the derivation for $s*$ for the $(s, Q)$ policy. The optimal conditions are found in Table 1. We create Table 3 from Table 1 by replacing

*s* with *S*, *Q* with *aR*, and *L* with *R* + *L*.

A similar table for discrete distributions can be constructed starting from Table 2.

Table 3. The ($R$, $S$) Policy for Continuous Distributions

| Situation | $C_S$ | Optimal reorder point | Normal solution |
|---|---|---|---|
| Fixed cost per stockout ($\pi_1$) | $\pi_1(1 - F(S))$ | $f_P(S^*) = \frac{hR}{\pi_1}$ | $\phi(k^*) = \frac{\sigma_P hR}{\pi_1}$ |
| Charge per unit Short ($\pi_2$) | $\pi_2 E_S$ | $F_P(S^*) = 1 - \frac{hR}{\pi_2}$ | $\phi(k^*) = 1 - \frac{hR}{\pi_2}$ |
| Charge per unit short per snit time ($\pi_3$) | $\pi_3 T_S$ | $E_S(S^*) = \frac{haR}{\pi_3}$ | $G(k^*) = \frac{haR}{\sigma_P \pi_3}$ |
| Charge per unit of lost sales ($\pi_L$) | $\pi_L E_S$ | $F(S^*) = 1 - \frac{hR}{hR + \pi_L}$ | $\phi(k^*) = 1 - \frac{hR}{hR + \pi_L}$ |

*Example 14*

We consider again the situation of Example 9 except the inventory is reviewed every month. The monthly demand for the product has a normal distribution with a mean of 100 and a standard deviation of 20. The holding cost is \$10 per unit per month. When it is necessary to backorder a customer request, the cost of paperwork and good will is estimated to be \$200 per unit. The lead time for orders is zero. Find the optimal inventory policy.

The parameters of the model are as follows

$R$ = 1 month

$h$ = \$10, the cost of holding one unit for one month

$\pi_2$ = 200, the cost of backordering one unit

$\mu$ = 100 and $\sigma$ = 20 during the review period (the distribution is normal)

With these values we can determine the optimal policy.

$$\phi(k^*) = 1 - \frac{hR}{\pi_2} = 1 - (10/200) = 0.95$$

Using the standard normal table, we find that the CDF has the value 0.957 for $z$ = 1.65. The order level is

$$S = 100 + 20(1.65) = 133.$$

The optimal policy is to order a quantity (133 – $x$) each month. The probability of a shortage during the month is

$$1 - \phi(k^*) = 0.05.$$

The safety stock is $S - \mu = 33$ units.

Now we consider the same problem when the review period is every 2 months. The situation is the same except

$R =$ 2 months

$\mu =$ 200 and $\sigma = 20\sqrt{2} = 28.28$ during the review period (here we assume that the monthly demands are independent)

$$\phi(k^*) = 1 - \frac{hR}{\pi_2} = 1 - (10)(2)/200) = 0.90.$$

Using a standard normal table, we find that the CDF has the value 0.899 for $k = 1.28$. Thus the reorder level is

$$S = 200 + 28.28(1.28) = 236.$$

The optimal policy is to order a quantity (236 – $x$) every two months. The safety stock is: $S - \mu = 36$. Three extra units of inventory are necessary for the longer review period. For the continuous review of Example 9 the safety stock is 17 units.

# 25.8 Exercises

**Section 25.2**

1. It is now January 1. A hardware distributor is reviewing his inventory policy for hammers, which have a relatively constant demand of 2000 units per month. The distributor buys the hammers from his supplier for $5 each and sells them for $10. Every time he places an order for replenishment a shipping and paper preparation cost of $500 is charged. The distributor's holding cost is 15% (annual) of his average investment in inventory. (The holding cost is $h$ = (0.15)(5) = $0.75 per hammer per year.)

   Answer each of the following independently ( one part does not depend on another).

   a. The current inventory policy is to replenish the inventory every month. What is the total annual cost of this policy?

   b. What is the optimal lot size when no shortages are allowed?

   c. The distributor wants to order only at the beginning of the month, so the cycle time must be in whole months. What is the optimal lot size in this case?

   d. An OR analyst suggests that money might be saved if the customers would accept backorders. If the distributor offers a discount of 1% of the purchase price for every day the customer has to wait for delivery, what is the optimal policy?

   e. The supplier, in an effort to get larger orders, offers a $0.10 discount on the price of hammers for an order of at least 10,000 units. Should the distributor take advantage of this deal? Assume no shortages are allowed.

   f. It is now January 1, and the current inventory is 3500 units. There is a lead time (time between when the order is placed and when it is delivered) of 1 month. When should the next order be placed for the solutions of parts (b), (c), (d), and (e) of this problem?

   g. Compute the annual profit on hammers for parts (a), (b), (c), (d), and (e).

2. Rather than an instantaneous replenishment of inventory as assumed in Section 25.2, the inventory is replaced by a production process that can add units to the inventory at a constant rate $r$ ($r > a$). Of course, the process cannot be operated continually but must be started and stopped. Each time the process is begun a setup cost $K$ is expended. Otherwise the inventory problem is the same as previously stated. Derive a formula for the optimal lot size when no shortages are allowed.

3. A meat market advertises that its hamburger is the freshest in town. To get rid of old stock, the price of the hamburger is reduced by $0.10 per pound for every hour it remains unsold after it is ground (for simplicity assume this reduction is linear in time). The demand for hamburger is 250 pounds per hour. No shortages are al-

lowed. The cost to set up and run the hamburger grinder is $100, independent of the amount processed. How much should be ground at each setup?

4. Consider a gift shop operator with a relatively constant demand for a certain type of wall hanging. She can order a case that holds 12 units for a cost of $1200 or a gross that holds 144 units for $13,000. These costs include delivery and product cost. The demand for the product is 12 units per month, so the alternatives represent ordering a 1-month or 1-year supply. As an approximation assume that demand is at a constant rate and no shortages are allowed. A principal concern for this business woman is the cost of capital, which is 20% per year. Which plan should she adopt?

5. Light bulbs come in cases of 144 bulbs. A case of bulbs costs $200. An office building uses bulbs at an average rate of 1000 bulbs per month. The company that sells the bulbs charges $50 per delivery, regardless of the number of cases delivered. The management of the building uses a 15% annual carrying cost rate for inventory. No shortages are allowed.

   a. Assuming the usage rate for bulbs is constant, how many cases of bulbs should be ordered in each delivery? Only whole cases can be ordered.

   b. With this order quantity, how often will bulbs be delivered?

   c. A long-standing policy has been to receive 10 cases per delivery. What is the annual cost penalty of this nonoptimal policy over the cost of the optimal policy?

6. Find the optimal policy for 5 if backorders are allowed. A failed bulb is simply not replaced until the beginning of the next month if the supply is exhausted. The management assumes a "loss of good will" charge of $10 per bulb per month for backorders.

**Section 25.4**

7. A garden nursery has a short selling season of 20 days for a certain kind of tree. The nursery currently has 20 such trees in stock and has one last opportunity to order more. The expected demand for trees is 3 per day, but the actual demand has a Poisson distribution. The nursery purchases the trees for $75 and sells them for $175. Trees remaining at the end of the season must be cared for until the next year. The cost of such care per tree is $20 and the holding cost on investment is $2. If demand occurs after the inventory is exhausted, the sales are lost. Find the number of trees that should be ordered and the probability that this inventory level will be sufficient to meet all demand. (Note that since leftover trees will actually be kept after the season and eventually sold, the purchase cost is not relevant. Only the shortage cost and holding cost should be used.)

8. For the parameters given in *Example 6* in the text, the probability of a shortage is almost 70%. Management has determined that this is too high. A level of 10% is considered an acceptable probability of shortage. If all other parameters remain the same, what penalty is implicitly being charged for a shortage? For this value find

the optimal policy assuming the distribution of demand is uniform between 50 and 250 items.

9. Use the penalty found in Exercise 8, and find the optimal policy if the demand is exponential distributed with a mean of 150.

10. A dishonest developer wants to make quick money building a condominium. He sets a 1-year time table for his activities, after which he will leave town for places unknown. His problem is to determine how many units to build. He estimates the following probabilities for the demand for his units. In the table below, $x$ is the number of units demanded and $p(x)$ is the probability of that demand.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $p(x)$ | 0.02 | 0.03 | 0.09 | 0.14 | 0.19 | 0.14 | 0.10 | 0.05 |
| $x$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 | |
| $p(x)$ | 0.05 | 0.05 | 0.04 | 0.04 | 0.03 | 0.02 | 0.01 | |

The selling price for a condo unit is $100,000, while it will cost only $40,000 to build it. If the developer runs out of units while there is still demand, he charges a lost opportunity cost of $10,000. If the year passes and he has not sold all his units, he will have to dump them for $20,000 each. How many units should he build to maximize his expected profit?

11. A military commander faces a dangerous mission which requires the use of helicopters. The commander wants to determine how many to bring on the mission. The helicopters cost $1 million each. At least five are required for mission success. Any number less than five is judged a serious detriment to the mission. In order to quantify the situation, the commander has attached a cost of $100 million for every unit less than five that finishes the mission. Any units that remain at the end of the mission are destroyed.

Failure-causing events are assumed to occur at random on the average of once every hour. The number of events is independent of the number of helicopters active. Each event will destroy one helicopter. The mission lasts 10 hours.
How many helicopters should the commander bring?

12. Consider a problem with the following parameters.

$$c = \$10,\ h = \$1,\ p = \$100, \text{ and } K = \$200.$$

Find the optimal $(s, S)$ policy when:

a. Demand has a uniform distribution with $A = 20$ and $B = 60$

b. Demand has an exponential distribution with a mean value of 40

**Section 25.5 - 25.7**

13. We repeat the conditions of Example 9. The demand per month has a normal distribution with mean 100 and standard deviation 20. The lead time is one week. Assume four weeks per month.

$a$ = 1200 units/year.

$h$ = \$120/unit-year.

$\pi_2$ = \$200.

$K$ = \$800.

Find the optimal ($s$, $Q$) policy when the following changes are made. The changes are not cumulative.

a. The average demand doubles but the standard deviation remains the same.

b. The standard deviation of demand per week doubles.

c. The fixed setup cost doubles.

d. The holding cost doubles.

e. The backorder cost doubles.

f. The lead time doubles.

14. Find the optimal order level for an inventory system for which orders are placed every 2 weeks. The weekly demand has a normal distribution with a mean of 25 and a standard deviation 5. The cost of the item is \$20. Interest per year is 20%. Assume a 50-week year. The backorder cost is \$5 per unit.

15. Use the data of Exercise 14 and add the information that the setup cost is \$200 and the lead time is 1 week. Find the optimal continuous review policy for the following situations.

a. The order quantity is set to the average demand for 1 month.

b. The reorder point is set at the average demand over the lead time.

c. Neither the reorder point nor the order quantity is specified.

16. A supplier of rebuilt engines expects an average of three engine customers every 2 months. Customer arrivals follow a Poisson process. The engines are obtained from a manufacturer who delivers 2 months after an order is placed.

a. What should be the supplier's reorder point if she wants a 60% chance of not having a shortage during the lead time?

b. What should be the reorder point if the lead time were 1 month?

17. Consider the light bulb situation of Exercise 5, except that demand is stochastic rather than deterministic. The weekly demand for light bulbs is a random variable with a normal distribution that has a mean of 250 bulbs and a standard deviation of 50 bulbs. An order for bulbs is placed once a month (on the first of the month) and the order is delivered right away. If the supply of bulbs runs out and a bulb fails, the tenant simply must wait until the beginning of the next month. For purposes of analysis, this event is assumed to have a cost of $10. Assume for simplicity that months are 4 weeks long and that there are 48 weeks in a year. What specific numerical rule should the management use to determine how much to order each month?

18. The average demand for a product at a warehouse is 1200 units per year. Customers arrive at random (a Poisson process) so the exact demand for any given period of time cannot be computed. The warehouse manager replenishes the inventory by a monthly order to the factory. The size of the order equals the previous month's sales.

    a. What kind of inventory system is this?

    b. What order level will provide a 95% chance that the inventory will not run out during an order cycle?

19. The manager in Exercise 18 installs computerized inventory control so that continuous review is possible. He adopts a ($s$, $Q$) policy. A lead time of 1 week passes between when an order is placed and when the inventory is replenished. A lot size of 100 is selected. Assume a month is 4 weeks.

    a. What reorder point will provide a 95% chance that the inventory will not run out during an order cycle?

    b. What is the average cycle time for this plan?

20. A gasoline distributor has a weekly demand that is approximately normally distributed. The average demand is 10,000 gallons per week; however, the standard deviation is 3000 gallons. His supply is replenished every 6 weeks. He must pay $0.75 per gallon. His cost of capital is 0.5% per week. He recovers this cost in the price he charges for gasoline, but if any remains in inventory when the next order arrives he charges a holding cost based on this interest rate and the value of the inventory. If the distributor runs out of supply during the period, he "borrows" gas from other distributors, paying an extra $0.10 per gallon for the privilege. The borrowed gas must be returned when his supply is replenished. What is the optimal inventory policy for the distributor?

21. One of the disadvantages of using large order quantities in a production process is, of course, large holding costs. Another disadvantage is the inflexibility associated with having a large inventory. If a design change is to be incorporated into the

product or some custom feature is to be added for particular customers, the modification must await the next production cycle while the entire inventory is depleted. The average length of the cycle is $Q/a$, so as $Q$ grows so does the length of the cycle. This might be called the lead time of the production process. The reason for a large order quantity is a large setup cost, so reducing setup cost results in reduced holding cost, decreased production lead time, and increased flexibility. This is one of the tenants of the "just-in-time" production systems, which go to great measures to reduce setup costs. Analyze the following production systems that are characterized by different setup costs $K$, and different annual costs to implement the production process $B$.

a. $K$ = \$10,000 , $B$ = \$10,000

b. $K$ = \$5,000 , $B$ = \$30,000

c. $K$ = \$1,000 , $B$ = \$110,000

In each case weekly demand follows a normal distribution with mean 50, and standard deviation 10. Use a 50-week year and an ($s$, $Q$) inventory policy. The holding cost is $h$ = \$200/unit-year. The lead time when a reorder is placed is 1 week. The shortage cost is \$100 per unit. Compute the optimal values of $s$ and $Q$ for each case and compare the total costs (including $B$) and production lead times. Choose the least cost system.

Table 4. Function Values for the Standard Normal Distribution, $y \in [-3,0]$ †

| $y$ | $\phi(y)$ | $\Phi(y)$ | $G(y)$ | $y$ | $\phi(y)$ | $\Phi(y)$ | $G(y)$ |
|---|---|---|---|---|---|---|---|
| -3.00 | 0.0044 | 0.0013 | 3.0004 | -1.50 | 0.1295 | 0.0668 | 1.5293 |
| -2.95 | 0.0051 | 0.0016 | 2.9505 | -1.45 | 0.1394 | 0.0735 | 1.4828 |
| -2.90 | 0.0060 | 0.0019 | 2.9005 | -1.40 | 0.1497 | 0.0808 | 1.4367 |
| -2.85 | 0.0069 | 0.0022 | 2.8506 | -1.35 | 0.1604 | 0.0885 | 1.3909 |
| -2.80 | 0.0079 | 0.0026 | 2.8008 | -1.30 | 0.1714 | 0.0968 | 1.3455 |
| -2.75 | 0.0091 | 0.0030 | 2.7509 | -1.25 | 0.1826 | 0.1056 | 1.3006 |
| -2.70 | 0.0104 | 0.0035 | 2.7011 | -1.20 | 0.1942 | 0.1151 | 1.2561 |
| -2.65 | 0.0119 | 0.0040 | 2.6512 | -1.15 | 0.2059 | 0.1251 | 1.2121 |
| -2.60 | 0.0136 | 0.0047 | 2.6015 | -1.10 | 0.2179 | 0.1357 | 1.1686 |
| -2.55 | 0.0154 | 0.0054 | 2.5517 | -1.05 | 0.2299 | 0.1469 | 1.1257 |
| -2.50 | 0.0175 | 0.0062 | 2.5020 | -1.00 | 0.2420 | 0.1587 | 1.0833 |
| -2.45 | 0.0198 | 0.0071 | 2.4523 | -0.95 | 0.2541 | 0.1711 | 1.0416 |
| -2.40 | 0.0224 | 0.0082 | 2.4027 | -0.90 | 0.2661 | 0.1841 | 1.0004 |
| -2.35 | 0.0252 | 0.0094 | 2.3532 | -0.85 | 0.2780 | 0.1977 | 0.9600 |
| -2.30 | 0.0283 | 0.0107 | 2.3037 | -0.80 | 0.2897 | 0.2119 | 0.9202 |
| -2.25 | 0.0317 | 0.0122 | 2.2542 | -0.75 | 0.3011 | 0.2266 | 0.8812 |
| -2.20 | 0.0355 | 0.0139 | 2.2049 | -0.70 | 0.3123 | 0.2420 | 0.8429 |
| -2.15 | 0.0396 | 0.0158 | 2.1556 | -0.65 | 0.3230 | 0.2578 | 0.8054 |
| -2.10 | 0.0440 | 0.0179 | 2.1065 | -0.60 | 0.3332 | 0.2743 | 0.7687 |
| -2.05 | 0.0488 | 0.0202 | 2.0574 | -0.55 | 0.3429 | 0.2912 | 0.7328 |
| -2.00 | 0.0540 | 0.0228 | 2.0085 | -0.50 | 0.3521 | 0.3085 | 0.6978 |
| -1.95 | 0.0596 | 0.0256 | 1.9597 | -0.45 | 0.3605 | 0.3264 | 0.6637 |
| -1.90 | 0.0656 | 0.0287 | 1.9111 | -0.40 | 0.3683 | 0.3446 | 0.6304 |
| -1.85 | 0.0721 | 0.0322 | 1.8626 | -0.35 | 0.3752 | 0.3632 | 0.5981 |
| -1.80 | 0.0790 | 0.0359 | 1.8143 | -0.30 | 0.3814 | 0.3821 | 0.5668 |
| -1.75 | 0.0863 | 0.0401 | 1.7662 | -0.25 | 0.3867 | 0.4013 | 0.5363 |
| -1.70 | 0.0940 | 0.0446 | 1.7183 | -0.20 | 0.3910 | 0.4207 | 0.5069 |
| -1.65 | 0.1023 | 0.0495 | 1.6706 | -0.15 | 0.3945 | 0.4404 | 0.4784 |
| -1.60 | 0.1109 | 0.0548 | 1.6232 | -0.10 | 0.3970 | 0.4602 | 0.4509 |
| -1.55 | 0.1200 | 0.0606 | 1.5761 | -0.05 | 0.3984 | 0.4801 | 0.4244 |
| -1.50 | 0.1295 | 0.0668 | 1.5293 | 0.00 | 0.3989 | 0.5000 | 0.3989 |

† $y$ is a standard normal variate
$\phi(y)$ is the probability density function, pdf
$\Phi(y)$ is the cumulative distribution function, CDF
$G(y) = \phi(y) - y[1 - \Phi(y)]$

Table 4. (Cont.) Function Values for the Standard Normal Distribution $y \quad [0,3]^{\dagger}$

| $y$ | $\phi(y)$ | $\Phi(y)$ | $G(y)$ | $y$ | $\phi(y)$ | $\Phi(y)$ | $G(y)$ |
|---|---|---|---|---|---|---|---|
| 0.00 | 0.3989 | 0.5000 | 0.3989 | 1.50 | 0.1295 | 0.9332 | 0.0293 |
| 0.05 | 0.3984 | 0.5199 | 0.3744 | 1.55 | 0.1200 | 0.9394 | 0.0261 |
| 0.10 | 0.3970 | 0.5398 | 0.3509 | 1.60 | 0.1109 | 0.9452 | 0.0232 |
| 0.15 | 0.3945 | 0.5596 | 0.3284 | 1.65 | 0.1023 | 0.9505 | 0.0206 |
| 0.20 | 0.3910 | 0.5793 | 0.3069 | 1.70 | 0.0940 | 0.9554 | 0.0183 |
| 0.25 | 0.3867 | 0.5987 | 0.2863 | 1.75 | 0.0863 | 0.9599 | 0.0162 |
| 0.30 | 0.3814 | 0.6179 | 0.2668 | 1.80 | 0.0790 | 0.9641 | 0.0143 |
| 0.35 | 0.3752 | 0.6368 | 0.2481 | 1.85 | 0.0721 | 0.9678 | 0.0126 |
| 0.40 | 0.3683 | 0.6554 | 0.2304 | 1.90 | 0.0656 | 0.9713 | 0.0111 |
| 0.45 | 0.3605 | 0.6736 | 0.2137 | 1.95 | 0.0596 | 0.9744 | 0.0097 |
| 0.50 | 0.3521 | 0.6915 | 0.1978 | 2.00 | 0.0540 | 0.9772 | 0.0085 |
| 0.55 | 0.3429 | 0.7088 | 0.1828 | 2.05 | 0.0488 | 0.9798 | 0.0074 |
| 0.60 | 0.3332 | 0.7257 | 0.1687 | 2.10 | 0.0440 | 0.9821 | 0.0065 |
| 0.65 | 0.3230 | 0.7422 | 0.1554 | 2.15 | 0.0396 | 0.9842 | 0.0056 |
| 0.70 | 0.3123 | 0.7580 | 0.1429 | 2.20 | 0.0355 | 0.9861 | 0.0049 |
| 0.75 | 0.3011 | 0.7734 | 0.1312 | 2.25 | 0.0317 | 0.9878 | 0.0042 |
| 0.80 | 0.2897 | 0.7881 | 0.1202 | 2.30 | 0.0283 | 0.9893 | 0.0037 |
| 0.85 | 0.2780 | 0.8023 | 0.1100 | 2.35 | 0.0252 | 0.9906 | 0.0032 |
| 0.90 | 0.2661 | 0.8159 | 0.1004 | 2.40 | 0.0224 | 0.9918 | 0.0027 |
| 0.95 | 0.2541 | 0.8289 | 0.0916 | 2.45 | 0.0198 | 0.9929 | 0.0023 |
| 1.00 | 0.2420 | 0.8413 | 0.0833 | 2.50 | 0.0175 | 0.9938 | 0.0020 |
| 1.05 | 0.2299 | 0.8531 | 0.0757 | 2.55 | 0.0154 | 0.9946 | 0.0017 |
| 1.10 | 0.2179 | 0.8643 | 0.0686 | 2.60 | 0.0136 | 0.9953 | 0.0015 |
| 1.15 | 0.2059 | 0.8749 | 0.0621 | 2.65 | 0.0119 | 0.9960 | 0.0012 |
| 1.20 | 0.1942 | 0.8849 | 0.0561 | 2.70 | 0.0104 | 0.9965 | 0.0011 |
| 1.25 | 0.1826 | 0.8944 | 0.0506 | 2.75 | 0.0091 | 0.9970 | 0.0009 |
| 1.30 | 0.1714 | 0.9032 | 0.0455 | 2.80 | 0.0079 | 0.9974 | 0.0008 |
| 1.35 | 0.1604 | 0.9115 | 0.0409 | 2.85 | 0.0069 | 0.9978 | 0.0006 |
| 1.40 | 0.1497 | 0.9192 | 0.0367 | 2.90 | 0.0060 | 0.9981 | 0.0005 |
| 1.45 | 0.1394 | 0.9265 | 0.0328 | 2.95 | 0.0051 | 0.9984 | 0.0005 |
| 1.50 | 0.1295 | 0.9332 | 0.0293 | 3.00 | 0.0044 | 0.9987 | 0.0004 |

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