Extra Problems for Chapter 3. Linear Programming Methods

20. (*Big-M Method*) An alternative to the two-phase method of finding an initial basic feasible solution by minimizing the sum of the artificial variables, is to solve a single linear program in which the objective function is augmented by a penalty term comprising the sum of the artificial variables. Assuming an artificial variable is introduced for each constraint, the objective function to be maximized is

$$z(M) = \sum_{j=1}^{n} c_j x_j - M \sum_{j=1}^{m} \alpha_i$$

where M is a "large" number.

a. When the solution to the original problem is finite, prove that the solution to the problem of maximizing $z(M)$ subject to the original set of constraints is identical to the former.

b. If the solution to the original problem is unbounded, how will this manifest itself in the solution to the modified problem with objective function $z(M)$?

c. If the solution to the original problem is infeasible, how will this manifest itself in the solution to the modified problem with objective function $z(M)$?

21. Solve the LP given in Exercise 19 using the big-M method discussed in Exercise 20.

22. Put the problem below into the simplex form by first multiplying each constraint by –1 and then adding slack variables. Let the slacks provide an initial basic solution. Solve using the dual simplex method.

$$\begin{aligned} \text{Minimize } z = \; & 3x_1 + 7x_2 + 4x_3 + 5x_4 \\ \text{subject to } \; & 2x_1 + x_2 + 3x_3 + 4x_4 \geq 7 \\ & x_1 + 2x_2 + 4x_3 + 2x_4 \geq 6 \\ & 3x_1 + 4x_2 + x_3 + x_4 \geq 5 \\ & x_j \geq 0, \; j = 1,\ldots,4 \end{aligned}$$

23. The optimal tableau for the following linear program is given below.

$$\begin{aligned} \text{Maximize } z = \; & 5x_1 - 3x_2 + 4x_3 - x_4 \\ \text{subject to } \; & 3x_1 + 4x_2 + 3x_3 - 5x_4 \leq 13 \\ & 5x_1 - 3x_2 + x_3 + 2x_4 \leq 15 \end{aligned}$$

$x_j \geq 0, \quad j = 1,\ldots,4$

Row	Basic	z	x_1	x_2	x_3	x_4	x_{s1}	x_{s2}	RHS
		Coefficients							
0	z	1	5.181	1.637	0	0	0.819	1.546	33.82
1	x_3	0	2.818	–0.636	1	0	0.182	0.455	9.18
2	x_4	0	1.091	–1.182	0	1	–0.091	0.273	2.91

a. If the right-hand sides of the original two constraints are changed to 20 and 5, respectively, the rightmost column of the tableau becomes (24.110, 5.915, –0.455). Write out the new tableau for this basic solution and use the dual simplex method to reoptimize.

b. Starting from the tableau given above, add the constraint, $x_1 + x_2 + x_3 + x_4 \leq 10$. Put the tableau into the simplex form and use the dual simplex method to find the new optimal solution.

24. Let u_j be an upper bound for x_j. It is always possible to solve an LP that contains simple variable upper bounds by adding the constraints $x_j \leq u_j$ to the model. This is inefficient, however, because each such constraint increases the size of the basis by one; the work required to solve an LP goes up accordingly. Fortunately, the simplex algorithm can be modified to account implicitly for simple upper bounds. This means that they don't have to be added to the tableau as a separate row.

a. When the *bounded variable simplex method* is used, variables at either their lower or upper bound are generally considered nonbasic. How should the rule used to select an entering variable be modified to account for a variable that is nonbasic at its upper bound. Hint: refer to the interpretation of the objective coefficients in Eq. (6) in Section 3.4 as marginal values.

b. How should the optimality test be modified to account for variables at their upper bound.

c. Modify Definition 5 regarding degeneracy when variable upper bounds are treated implicitly by the simplex method. Should Definition 8 be modified as well? Explain.

d. Explain how the ratio test used to determine the leaving variable has to be modified to account for simple upper bounds. Hint: there are three cases that must be considered.

25. Linear programming may be used to solve a relaxation of an integer program. The problems below represent binary integer programming models in which the requirement that the variables be *either* 0 or 1 has been replaced with the less restrictive requirement that the variables be *between* 0 and 1. The bounded variable simplex algorithm is particularly useful for solving this kind of problem since all the structural variables are bounded by 1.

a. Referring to the modifications suggested in Exercise 24, solve the following linear program with the bounded variable simplex method.

$$\text{Maximize } 84x_1 + 48x_2 + 25x_3 + 29x_4 + 128x_5$$

$$\text{subject to } 49x_1 + 30x_2 + 19x_3 + 29x_4 + 91x_5 \le 130$$

$$0 \le x_j \le 1, \quad j = 1,\ldots,5$$

b. *Continuous knapsack problem.* Develop a simple algorithm for solving the single constraint linear program

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_j x_j \le b$$

$$0 \le x_j \le 1, \quad j = 1,\ldots,n$$

Solve the linear program in part (a) with your algorithm.

26. For the problem below, construct the tableau with the following information:

(i) x_{s1}, x_{s2} and x_{s3} are the slack variables for the three constraints,

(ii) x_2, x_4, x_{s2} and x_{s3} are nonbasic,

(iii) x_1, x_3 and x_{s1} are basic,

(iv) upper bounds on x_j are treated implicitly.

Manipulate the tableau to obtain the simplex form and use the bounded variable simplex technique mentioned in Exercise 24 to obtain the optimal solution.

$$\text{Minimize } 3x_1 + 7x_2 + 4x_3 + 5x_4$$

$$\text{subject to } 2x_1 + x_2 + 3x_3 + 4x_4 \ge 6$$

$$x_1 + 2x_2 + 4x_3 + 2x_4 \ge 6$$

$$3x_1 + 4x_2 + x_3 + x_4 \ge 6$$

$$0 \le x_j \le 1, \; j = 1, 2, 3, 4$$

31. Starting form the tableau given below, use the usual rules for selecting the entering and leaving variables, and perform two iterations of the simplex method. Show the corresponding tableaus in their entirety.

Row	Basic	Coefficients						RHS
		z	x_1	x_2	x_3	x_4	x_5	
0	z	1	0	−3	0	0	−5	20
1	x_3	0	0	2	1	0	4	10
2	x_1	0	1	−1	0	0	2	0
3	x_4	0	0	3	0	1	0	15

32. From the tableau given in Exercise 31, determine the original formulation of the problem, assuming that x_3, x_4 and x_5 are slack variables.

34. Solve the linear program below to completion by hand. Comment on any characteristics the final solution may possess.

$$\text{Maximize } z = 2x_1 - x_2 + 3x_3 + 5x_4$$

$$\text{subject to } \quad x_1 - x_2 - 3x_3 \leq 10$$

$$-2x_1 + x_2 - 4x_4 \leq 5$$

$$5x_3 + 4x_4 \leq 20$$

$$x_j \geq 0, \quad j = 1,\ldots,4$$

40. A linear program is given below along with the optimal tableau.

$$\text{Maximize } z = 5x_1 - 3x_2 + 4x_3 - x_4$$

$$\text{subject to } \quad 3x_1 + 4x_2 + 3x_3 - 5x_4 \leq 13$$

$$5x_1 - 3x_2 + x_3 + 2x_4 \leq 15$$

$$x_j \geq 0, \quad j = 1,\ldots,4$$

Row	Basic	Coefficients z	x_1	x_2	x_3	x_4	x_{s1}	x_{s2}	RHS
0	z	1	5.181	1.637	0	0	0.819	1.546	33.82
1	x_3	0	2.818	–0.636	1	0	0.182	0.455	9.18
2	x_4	0	1.091	–1.182	0	1	–0.091	0.273	2.91

a. For this solution give the matrices $\mathbf{B}$, $\mathbf{B}^{-1}$, $\mathbf{x}_B$, $\mathbf{c}_B$, and $\mathbf{b}$.

b. What are the primal and dual solutions given by the tableau?

c. Which constraints are tight and which are loose for the optimal solution?

41. In each case below, say as much as you can about the basic solution associated with the given data.

a. $\mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2.5 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 8 \\ 6 \\ 5 \end{pmatrix}$

b. $\mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2.5 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix}$

42. What can you say about the *next* basic solution when given the information below? Assume that the entering variable has a negative reduced cost.

a. $\mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2.5 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \\ 14 \end{pmatrix}$, entering column $\mathbf{A}_s = \begin{pmatrix} 1 \\ -1 \\ -4.5 \end{pmatrix}$

b. $\mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2.5 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \\ 14 \end{pmatrix}$, entering column $\mathbf{A}_s = \begin{pmatrix} 2 \\ 1 \\ -1.25 \end{pmatrix}$

43. Consider the following linear program.

Maximize $z = 2x_1 + 3x_2 + x_3 + 4x_4$

subject to $x_1 - x_3 + x_4 \geq 5$

$$-x_1 + 2x_2 \qquad + x_4 \le 6$$

$$x_2 + 2x_3 + 0.5x_4 \le 8$$

$$x_j \ge 0, \quad j = 1,\ldots,4$$

Let the slacks be variables x_5 through x_7, and let x_α be the artificial variable introduced for the first constraint. The problem is to be solved with the revised simplex algorithm coupled with the two-phase method. Perform two complete iterations of the algorithm using the "most negative reduced cost" rule to determine the variable to enter the basis. Show the basis inverse after each iteration.

44. You are given the following linear program.

$$\begin{aligned} \text{Maximize } z = {} & x_1 + 2x_2 + 3x_3 \\ \text{subject to } \quad & x_1 + 2x_2 + 3x_3 \ge 6 \\ & 3x_1 + 2x_2 + x_3 \ge 6 \\ & x_1 + x_2 + x_3 \le 10 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \end{aligned}$$

a. Let x_4, x_5 and x_8 be the slack variables for the three constraints, and let x_6 and x_7 be the artificial variables for the first two constraints. Write out the problem that is to be solved in phase 1.

b. Perform the first iteration of the revised simplex method for the problem defined in part (a). Complete the iteration through the pivot operation that shows the new basis inverse.

c. At the end of phase 1, the basic variables are (in this order) x_3, x_1 and x_8 (the slack variable for the third constraint). The corresponding basis and basis inverse are shown below. Perform the next revised simplex iteration appropriate for this problem. Complete the iteration through the pivot operation that shows the new basis inverse and calculate the new BFS.

$$\mathbf{B} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{B}^{-1} = \begin{pmatrix} 3/8 & -1/8 & 0 \\ -1/8 & 3/8 & 0 \\ -1/4 & -1/4 & 1 \end{pmatrix}$$

d. At optimality x_3, x_4 and x_5 are the basic variables. The corresponding basis and basis inverse are shown below. What are the primal and dual optimal solutions?

$$\mathbf{B} = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{B}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & 1 \end{pmatrix}$$

e. Using the solution implied by the information in part (d), change the RHS of constraint 3 in the original problem to 6. Perform one iteration of the dual simplex algorithm in an attempt to make the solution feasible. Go so far as to determine the values for the variables in the next basic solution. Has feasibility been obtained?

45. (*Simple Upper Bounds*) The problem below has one constraint with simple upper bounds on the variables.

$$\text{Maximize } z = 84x_1 + 35x_2 + 25x_3 + 29x_4 + 128x_5$$

$$\text{subject to } 50x_1 + 30x_2 + 20x_3 + 30x_4 + 100x_5 \le 90$$

$$0 \le x_j \le 1, \quad j = 1,\ldots,5$$

Consider the solution with x_1 and x_2 at their upper bounds, x_4, x_5 and x_6 (the slack variable) at zero, and x_3 basic. For the simplex method modified to accommodate simple upper bounds as discussed in Exercise 24, find the following for this solution.

a. The basis matrix.

b. The basis inverse.

c. The dual variables.

d. The reduced costs.

e. From the reduced costs, tell which variables are candidates to enter the basis.

f. If x_1 is allowed to decrease from its upper bound, does x_3 increase or decrease? How much can x_1 decrease?