Equivalent Linear Programs

There are a number of problems that do not appear at first to be candidates for linear programming (LP) but, in fact, have an equivalent or approximate representation that fits the LP framework. In these instances, the solution to the equivalent problem gives the solution to the original problem. This appendix describes the transformations that can be used to convert a nonlinear problem to a linear program for the following three situations: (i) the objective is to maximize a separable, concave nonlinear function; (ii) the objective is to maximize the minimum of a set of linear functions; and (iii) there are several prioritized objectives with specified goals.

A.1 Nonlinear Objective Function

In some cases, linear programming can be used even when nonlinear terms are present. Consider the following mathematical programming model in compact algebraic form.

$$\text{Maximize } z = \sum_{j=1}^{n} f_j(x_j)$$

$$\text{subject to } \quad \sum_{j=1}^{n} a_{ij}x_j \le b_i, \quad i = 1,\ldots,m$$

$$x_j \ge 0, \quad j = 1,\ldots,n$$

The m constraints are linear but the objective consists of n nonlinear, separable terms $f_j(x_j)$, each a function of a single variable only. When the objective function can be written in this manner and each $f_j(x_j)$ is concave (see below), the above maximization problem may be approximated with a linear model and solved with a linear programming algorithm. When the functions $f_j(x_j)$ are not all concave this approach will not work. The absence of concavity requires the development of an integer programming model, as described in Chapter 8.

An analogous situation exists when the objective is to minimize a separable, convex function. An approximate linear programming model can be developed, but similarly, minimizing a concave function requires the use of integer variables.

Concave Functions

We first consider the case in which the function $f_j(x_j)$ is concave. The solid line in Fig. 1 depicts the graph of such a function. Later in the book the definition of concavity will be made precise but for now we say that a concave function has the characteristic that a straight line drawn between any two points on its graph falls entirely on or below the graph. A function that has a continuous first derivative is concave if the second derivative is everywhere nonpositive.

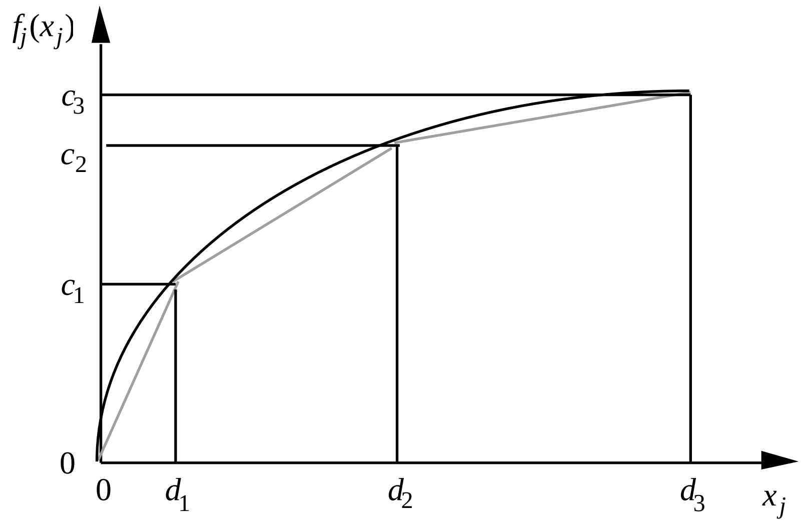

Figure 1. Concave function with a piecewise linear approximation

The dotted line in Fig. 1 represents a piecewise linear approximation to the concave function. The approximation identifies r break points along the x_j axis: $d_1, d_2, \ldots, d_r$, and r corresponding points along the f_j axis: $c_1, c_2, \ldots, c_r$. Now we have r pieces representing the objective function with the first starting at the origin. If $f_j(0)$ does not equal 0, the function $f_j(x_j)$ can be replaced by $f_j(x_j) - f_j(0)$ without affecting the optimal solution so we can always put $d_0 = c_0 = 0$.

The piecewise linear approximation is implemented in a linear programming model by defining new variables $x_{j1}, x_{j2}, \ldots, x_{jr}$ to represent the pieces. The slope of the kth segment is

$$s_{jk} = (c_k - c_{k-1})/(d_k - d_{k-1})$$

The piecewise linear approximation to the jth term in the objective function is

$$f_j(x_j) \leftarrow \sum_{k=1}^{r} s_{jk} x_{jk} .$$

In each constraint, the variable x_j is replaced by

$$x_j \leftarrow \sum_{k=1}^{r} x_{jk}$$

and the new variables must satisfy the following bounds

$$0 \le x_{jk} \le d_k - d_{k-1}, \ \ k = 1,\ldots,r$$

Of course, when $f_j(x_j) = c_j x_j$ no substitution is necessary; otherwise, each original variable x_j must be replaced with r_j new variables. Here r_j is the number of break points along the x_j-axis and may vary from one variable to the next. When the appropriate substitutions are made, the approximate model becomes

$$\text{Maximize } \ z = \sum_{j=1}^{n} \sum_{k=1}^{r_j} s_{jk} x_{jk}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \sum_{k=1}^{r_j} a_{ij} x_{jk} \le b_i, \quad i = 1,\ldots,m$$

$$0 \le x_{jk} \le d_k - d_{k-1}, \quad k = 1,\ldots,r_j, \ \ j = 1,\ldots,n$$

Thus to obtain a linear model, one pays the price in terms of increased problem size. The approximation can be made as accurate as desired by defining enough break points but with a corresponding increase in dimensionality.

The only remaining issue in the linearization process is whether a solution to the new problem is equivalent to a solution to the original problem. For x_j replaced by $\Sigma_{k=1}^{r} x_{jk}$, how can we be sure that the pieces of the approximation will be included in the solution in the proper order? Evidently, if x_{jk} is greater than 0, the solution will not be valid unless the variables for all the preceding pieces x_{jl}, where $l < k$, are at their upper bounds. There are no explicit constraints in the model to guarantee this.

Fortunately, when we are maximizing and the individual functions $f_j(x_j)$ are concave, the variables will enter in the proper order without explicit constraints. This is because the objective coefficients s_{jk} are decreasing with k. The goal of maximization will cause the pieces with the greatest slope to be selected first as the associated x_{jk} variables take on positive values.

Convex Functions

Figure 2 depicts a typical convex function which, by definition, has the property that a straight line drawn between any two points on its graph lies entirely on or above the graph. A function that has continuous first and second derivatives is convex if the second derivative is everywhere nonnegative. In general, if $f_j(x_j)$ is a convex function then $-f_j(x_j)$ is a concave function. One is a reflection of the other around the horizontal axis.

When the goal is to maximize the objective, the pieces of the linearized function enter the solution in exactly the reverse of the proper order. In this case, supplementary binary variables must be used to enforce the correct sequence so the resultant model would no longer satisfy the linear programming assumptions.

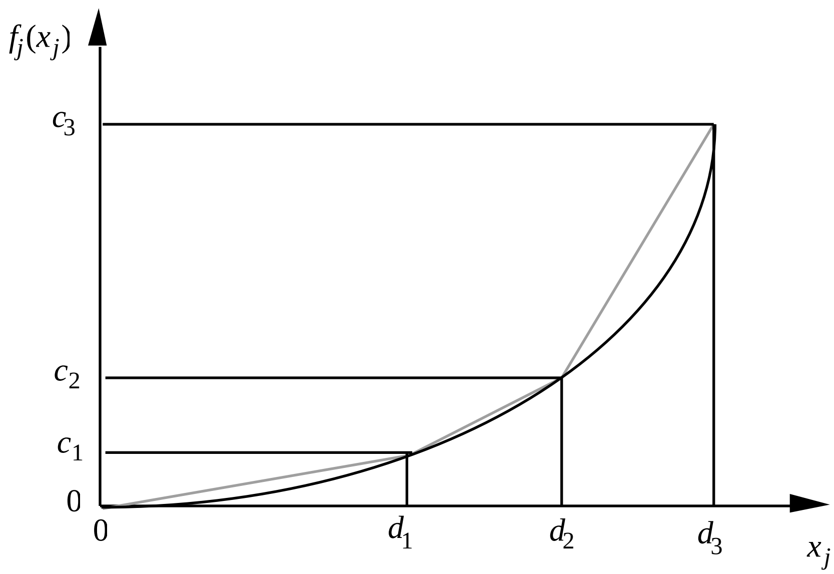

Figure 2. Piecewise linear approximation of convex function

Minimization Problems

A simple but important observation in optimization worth repeating is that minimizing a function is equivalent to maximizing the same function with its sign reversed. Because a convex term, $f_j(x_j)$, becomes concave when it is negated, we can conclude that a minimization problem with convex, separable terms in the objective can be approximated in the same way as a concave function when the goal is to maximize. Reasoning as before, however, when concave terms appear in a minimization objective, linear programming cannot be used.

Modeling

When maximizing profit, revenue terms in the objective function have a positive sign and cost terms have a negative sign. Thus in order to use linear programming to find a solution, all revenue terms must be concave functions and all cost terms must be convex functions. There are important practical instances where this is evident. The revenue or benefit received from the sale of most commodities has a concave shape because of the principle of decreasing marginal returns. There are many examples of price discounts to gain additional sales, as illustrated by airline ticket discount plans. On the other hand, cost functions are often convex with respect to the quantities produced or purchased. For example, to increase output in the near term, it may be necessary to pay overtime or use less efficient means of production. These circumstances promise an easy solution to the problem because the model can be approximated with piecewise linear terms and solved with a linear programming algorithm.

Unfortunately, there are also many practical situations when linear programming cannot be used. These most often arise in problems involving capacity expansion of facilities. The cost of building and operating a facility commonly involves economies of scale; the larger the facility, the smaller the marginal cost. This relationship implies a concave cost function that cannot be approximated with a linear programming model. To obtain a solution, it would be necessary to develop a model that used piecewise linear approximations as well as integer variables. The computational effort to solve such a model would be significantly greater than that required to solve a standard linear program.

Example

The problem below involves the maximization of a concave, separable quadratic function over a set of linear constraints. We use a piecewise linear approximation for the nonlinear terms in the objective to develop an LP model.

$$
\begin{aligned}
\text{Maximize } z &= -x_1^2 - 2x_2^2 + 8x_1 + 16x_2 \\
\text{subject to } \quad & 0.9x_1 + 1.2x_2 \le 5 \\
& x_1 \le 3 \\
& x_1 \ge 0,\; x_2 \ge 0
\end{aligned}
$$

The separable nonlinear terms of the objective are:

$$f_1(x_1) = -x_1^2 + 8x_1$$

$$f_2(x_2) = -2x_2^2 + 16x_2$$

We choose to use a piecewise approximation with the integers as breakpoints. Since the second constraint indicates that the upper bound on x_1 is 3 and first constraint implies that the upper bound on x_2 is 4, we have r_1 = 3 and r_2 = 4 giving the following breakpoints:

x_1 breakpoints

k	0	1	2	3
d_k	0	1	2	3
c_k	0	7	12	15
s_{1k}	—	7	5	3

x_2 breakpoints

k	0	1	2	3	4
d_k	0	1	2	3	4
c_k	0	14	24	30	32
s_{2k}	—	14	10	6	2

The linear programming approximation is then

$$\begin{array}{llllllllll}
\text{Maximize} & z = 7x_{11} & + 5x_{12} & + 3x_{13} & + 14x_{21} & + 10x_{22} & + 6x_{23} & + 2x_{24} & & \\
\text{subject to} & 0.9x_{11} & + 0.9x_{12} & + 0.9x_{13} & + 1.2x_{21} & + 1.2x_{22} & + 1.2x_{23} & + 1.2x_{24} & \le & 5 \\
 & x_{11} & + x_{12} & + x_{13} & & & & & \le & 3 \\
\end{array}$$

$$0 \le x_{ik} \le 1 \text{ all } i \text{ and } k$$

Because all the nonlinear terms are concave and the objective is to maximize, the solution to the LP model will yield a valid approximation to the solution of the original model. Using our Excel add-ins, we find that the optimum is

$$x_{11} = x_{12} = 1,\ \ x_{13} = 0,\ \ x_{21} = x_{22} = 1,\ \ x_{23} = 0.667,\ \ x_{24} = 0$$

Translating this solution into terms of the original problem gives

$$x_1 = 2,\ \ x_2 = 2.667.$$

A.2 Maximizing the Minimum

There are a variety of situations where maximizing total profit or minimizing total cost may not be the preferred course of action. When resources have to be distributed over more than a single entity or organization, the goal might be to maximize the minimum profit that is realized by any of the entities. Similarly, it might be optimal to set policy so that the maximum cost that is incurred by any of the organizations is minimized. In these scenarios the decision maker is implicitly hedging against the worst possible outcome by specifying what is respectively called a *maximin* or *minimax* strategy. This type of worst-case analysis is common when the decision maker is faced with an uncertain outcome and is risk averse.

A global optimum has no concern for fairness so one entity may be treated poorly in comparison to another when such a solution is implemented. For example, the best locations for fire stations may not be the solution that minimizes total response time, but rather the one that minimizes the maximum response time over a number of neighborhoods. One objective for fitting a line to a set of points is to minimize the total deviation between the points and the line. A reasonable alternative would be to minimize the maximum deviation over the set of points. Similarly, the corporate problem of allocating resources to decentralized divisions could be solved to maximize total profit; however, an alternative approach would be to maximize the minimum profit of the divisions. For problems whose constraints are otherwise linear, this kind of objective can be modeled as linear program.

Maximin Objective

Let us assume that we have a linear programming model defined by a set of constraints and t objective functions of the following form.

$$z_k = c_{0k} + c_{1k}x_1 + c_{2k}x_2 + \cdots + c_{nk}x_n$$

$$= c_{0k} + \sum_{j=1}^{n} c_{jk}x_j, \; k = 1,\ldots,t$$

Each function z_k is a hyperplane. The goal is to find an n-dimensional vector $\mathbf{x} = (x_1,\ldots,x_n)$ that minimizes the function $f(\mathbf{x})$ given by

$$f(\mathbf{x}) = \text{minimum}\left\{c_{01} + \sum_{j=1}^{n} c_{j1}x_j,\, c_{02} + \sum_{j=1}^{n} c_{j2}x_j, \ldots, c_{0t} + \sum_{j=1}^{n} c_{jt}x_j\right\}$$

subject to the constraints of the problem. Although $f(\mathbf{x})$ is not given in explicit form, it can be evaluated easily from the above equation for any real value of $\mathbf{x}$.

A useful property of $f(\mathbf{x})$ is that it is piecewise linear and concave. This is illustrated in Fig. 3 for $t = 4$ and x a scalar. The implication is that the problem of maximizing $f(\mathbf{x})$ subject to linear constraints can be transformed into a linear programming, even though $f(\mathbf{x})$ is not separable.

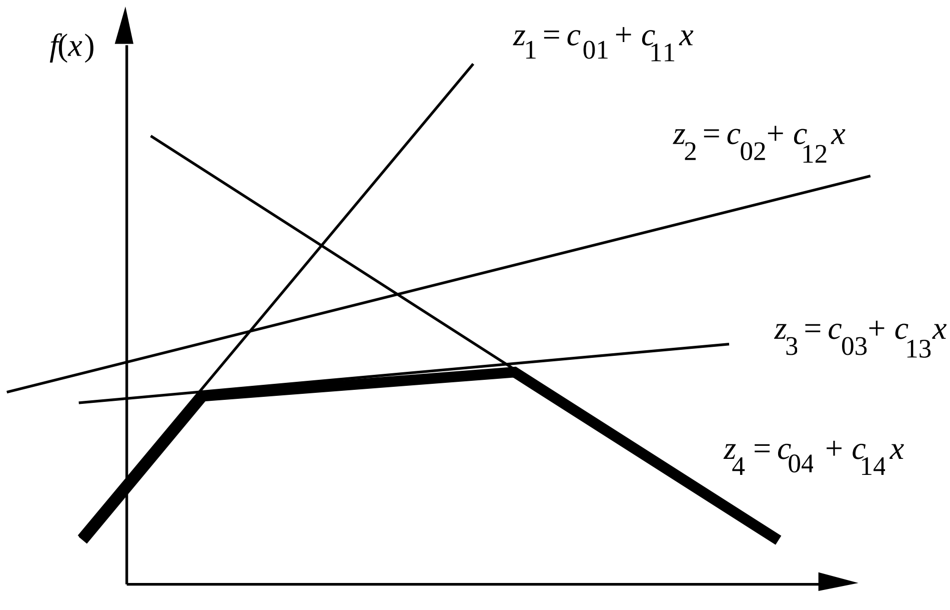

Figure 3. Function defined by minimum of several hyperplanes

The transformation is based on the observation that $f(\mathbf{x})$ is equal to the smallest number, call it z, that satisfies $z \geq c_{0k} + c_{1k}x_1 + \cdots + c_{nk}x_n$ for all k. The equivalent optimization problem is then

$$\text{Maximize } z$$

$$\text{subject to } \sum_{j=1}^{n} c_{jk}x_j \geq z, \quad k = 1,\ldots,t \qquad (\mathrm{M}_k)$$

plus the original linear constraints. The decision variables are z and x_j, $j = 1,\ldots,n$, which may or may not be restricted to be nonnegative. Thus in the transformation we have introduced one new variable z to be maximized, and t additional linear constraints: (M_1), (M_2),…,(M_t).

Minimax Objective

Reversing of the above case gives the objective to be minimized as

$$\text{Maximum } \{z_1, z_2, \ldots, z_t\}.$$

The equivalent linear program is

$$\text{Minimize } z$$

$$\text{subject to} \quad \sum_{j=1}^{n} c_{jk} x_j \le z, \quad k = 1,\ldots,t \qquad (M_k)$$

plus the original constraints of the problem. The added constraints force all the linear functions, z_k, to be less than or equal to the variable z that is to be minimized.

A One Dimensional Location Problem

At a prominent Texas university, the various engineering departments are located in buildings along a single street, as shown in Fig. 4. The distances given in the figure are in feet from the assumed origin. The dean of engineering wants to locate his office somewhere along the street. All locations are allowed (i.e., the street can be considered as a continuum of possible locations). The weekly number of trips by faculty and others between the Dean's office and the departments are listed in the trip table below.

The following three optimization criteria are being considered.

a. Minimize the total distance traveled.

b. Minimize the maximum distance traveled from any of the departments.

c. Minimize the total distance traveled, but no department is to be more than 300 feet from the dean's office.

Figure 4. Map showing the location of engineering departments

Trip table

Department, i	CE	EE	IE	PE	CHE	ME
Trips, w_i	137	160	15	76	52	125

Solution Idea

Each of the three problems will be solved in turn. To begin, define the location of the dean's office as the decision variable x. The distance from this office to a department is computed by considering whether it is to the left or to the right of the department. For the PE department the appropriate equation is

$$x + L_{\text{PE}} - R_{\text{PE}} = 350$$

where L_{PE} is the distance that the dean's office is to the left of PE and R_{PE} is the distance the office is to the right of PE. The distance between the office and the PE department is

$$L_{\text{PE}} + R_{\text{PE}}.$$

Logically, at most one of these variables will be positive in any solution.

Formal Model

VARIABLE DEFINITIONS

x = coordinate of the dean's office

L_i = distance from dean's office to department i when the office is to the left

R_i = distance from dean's office to department i when the office is to the right

Let subscript $i = 1,\ldots,6$, have the following association:

1 = CE, 2 = EE, 3 = IE, 4 = PE, 5 = CHE, 6 = ME.

CONSTRAINTS DEFINING DISTANCES

$$x + L_i - R_i = a_i \quad \text{for } i = 1,\ldots,6$$

where a_i is the x-coordinate of department i as shown in Fig. 4.

TOTAL DISTANCE TRAVELED

$$D = 137(L_1 + R_1) + 160(L_2 + R_2) + \cdots + 125(L_6 + R_6)$$

The distance traveled, D, is a function of the left and right variables, L_i and R_i. It is determined by weighting each pair by the corresponding values w_i in the trip table above, and then summing each term. In an optimal solution, at most one of the variables in the pair (L_i,R_i) will be positive.

Criterion a. The goal here is to minimize the single objective of total distance traveled. This leads to the following optimization problem.

$$\text{Minimize } D = \sum_{i=1}^{6} w_i(L_i + R_i)$$

$$\text{subject to } x + L_i - R_i = a_i, \quad i = 1,\ldots,6$$

$$x \geq 0, \; L_i \geq 0, \; R_i \geq 0, \quad i = 1,\ldots,6$$

Solving gives the optimal location $x^* = 150$ with the minimum total distance traveled $D^* = 86{,}100$.

Criterion b. We now wish to minimize the maximum distance traveled from any of the departments. Let v denote this distance with the stipulation that $L_i + R_i \leq v$ for $i = 1,\ldots,6$. The optimization problem is

$$\text{Minimize } v$$

$$\text{subject to } x + L_i - R_i = a_i, \quad i = 1,\ldots,6$$

$$L_i + R_i - v \leq 0, \quad i = 1,\ldots,6$$

$$x \geq 0, \; L_i \geq 0, \; R_i \geq 0, \quad i = 1,\ldots,6$$

The optimal location is $x^* = 300$ with $v^* = 225$, and total distance $D^* = 95{,}325$. The revised criterion has caused the total distance to increase.

Criterion c. For this part we use the constraints of part (b) and the objective of part (a) with the additional stipulation that

$$v \leq 300$$

Minimizing total distance gives $x^* = 225$, $D^* = 89{,}025$, and $v^* = 300$. By specifying a goal for the maximum distance, we have obtained an intermediate value for total distance.

A.3 Goal Programming

Most decision-making situations do not proceed from a single point of view or admit a single objective. In fact, many decisions must be made in the face of competing interests in a confrontational environment. Consider a local zoning commission that must balance the desires of residents, small businesses, developers, and environmentalists; or a corporate manager who must allocate the annual budget to several operating divisions; or a recent college graduate who must weigh salary, location, work environment, and fringe benefits of several job offers. The problem of the decision maker is to balance the goals of the competitors in such a way that most are to some extent satisfied; in other words, to reach a compromise. Administrators are always looking for the perfect compromise, the one that satisfies everyone, but of course this is rarely found. Rather we rely on committees, commissions, elections, contests, and even chance as ways to arrive at decisions that, at best, only partially satisfy the participants.

Up until this point in the chapter, the models presented have been limited to a single optimization criterion. The methods of goal programming extend our modeling capabilities by offering ways to deal with more than one objective at a time. They do not, however, provide the complete answer because whenever there are competing goals, it is difficult if not impossible to conduct a purely objective analysis that yields the "best" decision. There will always be a subjective component in the analysis that reflects the decision maker's preferences. Nevertheless, the goal programming approach does provide an organized way of considering more than one objective at a time and often yields compromise solutions that are acceptable to the protagonists. The basic idea is to establish specific numeric goals for each objective, and then to seek a solution that satisfies all the given constraints while minimizing the sum of deviations from the stated goals. Frequently, the deviations are weighted to reflect the relative importance of each objective function.

Definitions

Objective function ($f_k(\mathbf{x})$): One of several functions of the decision variables, $\mathbf{x} = (x_1, \ldots, x_n)$, that evaluates the attainment of some measure of effectiveness. Only linear functions are considered here.

$$f_k(\mathbf{x}) = c_{1k}x_1 + c_{2k}x_2 + \cdots + c_{nk}x_n = \sum_{j=1}^{n} c_{jk}x_j, \quad k = 1,\ldots,t$$

Lower One-Sided Goal: For the kth objective function, a lower limit, L_k, that the decision maker does not want to fall below. It is desired to achieve a value of "at least" L_k for the objective. Exceeding this value is permissible. The goal might be written as a "greater than or equal to" constraint:

$$f_k(\mathbf{x}) = \sum_{j=1}^{n} c_{jk}x_j \geq L_k$$

In the goal programming methodology, this is not a hard constraint so we allow solutions $\mathbf{x}$ such that $f_k(\mathbf{x}) \leq L_k$.

Upper One-Sided Goal: For the kth objective function, an upper limit, U_k, that the decision maker does not want to exceed. It is desired to achieve a value of "at most" U_k for the objective. The goal might be written as a "less than or equal to" constraint:

$$f_k(\mathbf{x}) = \sum_{j=1}^{n} c_{jk}x_j \leq U_k$$

Again, this is not a hard constraint so we allow solutions that yield values of $f_k(\mathbf{x})$ that exceed U_k if these lead to the best compromise.

Two-Sided Goal: Sets a specific target value, G_k, for the kth objective so the value of $f_k(\mathbf{x})$ should be "equal to" some G_k. The goal might be written as an equality constraint:

$$f_k(\mathbf{x}) = \sum_{j=1}^{n} c_{jk}x_j = G_k$$

The solution process will allow for deviations from this goal in either direction.

A particular objective will usually appear as either a lower one-sided, upper one-sided, or two-sided goal. In some cases, though, both upper and lower goals may be specified for an objective with the range between them defining a region of indifference.

Goal Constraint: The central construct in goal programming is the deviation variable. Let

$y_k^+ =$ positive deviation or the amount by which the kth goal is exceeded

$y_k^- =$ negative deviation or the amount by which the kth goal is underachieved

One of the following three constraints is used in the linear programming model to measure the deviation from the goal.

1. $\sum_{j=1}^{n} c_{jk}x_j - y_k^+ + y_k^- = L_k$

2. $\sum_{j=1}^{n} c_{jk}x_j - y_k^+ + y_k^- = U_k$

3. $\sum_{j=1}^{n} c_{jk}x_j - y_k^+ + y_k^- = G_k$

As can be seen, each of these constraints has the same form. This might seem odd at first but should become clear when we explain how objective functions are constructed. If more than one constraint were to be used in a model, say, to define a region of indifference, it would be necessary to distinguish each pair of deviation variables.

Penalty Weights (p_k^+ and p_k^-): Constants that measure the per unit penalty for violating goal constraint k . Let

p_k^+ = the penalty applied to the positive component.

p_k^- = the penalty applied to the negative component.

The three kinds of goals are associated with the following penalty assignments:

lower one-sided goal: $p_k^- > 0,\ p_k^+ = 0,$

upper one-sided goal: $p_k^- = 0,\ p_k^+ > 0,$

two-sided goal: $p_k^- > 0,\ p_k^+ > 0.$

When a lower bound is specified for the kth objective, for example, we set $p_k^- > 0$ because we want to penalize the underachievement of the goal L_k. We don't want to penalize its achievement, though, so we set $p_k^+ = 0$. Similar reasoning applies to the other two types of penalty assignments. In the goal programming model, the function to be optimized comprises terms of the form $z_k = p_k^+y_k^+ + p_k^-y_k^-$.

Nonpreemptive Goal Programming: In this approach, we put all the goals in the objective function and solve the linear program a single time. The objective for the problem is the weighted sum of the deviation variables.

The penalties measure the relative importance of the goals. The objective is to

$$\text{Minimize } z = \sum_{k=1}^{n} z_k = \sum_{k=1}^{n} \left(p_k^+ y_k^+ + p_k^- y_k^- \right)$$

Because the goals very often are measured on different scales, the penalties play the double role of transforming all goals to the same dimensional units as well as specifying their relative importance. In this approach, the subjective step is the determination of the weights. Different weights will often yield very different solutions.

Preemptive Goal Programming: Here the goals are divided into sets and each set is given a priority; i.e., first, second, and so on. The assumption is that a higher priority goal is absolutely more important than a lower priority goal. The solution is obtained by initially optimizing with respect to the first-priority goals without regard to the values of lower priority objectives. Then, holding constant the value of the first-priority objective function by adding the constraint $z_1(y_1^+, y_1^+) = z_1^*$, the optimal solution is obtained for the second-priority goals. The feasible solution space for this second problem is the set of alternate optima for the first problem.

The process continues until all priorities are considered. If no alternate optima exist at the end of a particular stage, we have reached the end of the computations so we must be satisfied with the current values of the lower priority objectives. If several goals have about the same priority we include all them in the set in the objective at the appropriate step of the process. The relative importance of the goals within any set are reflected by the specification of the penalty weights, as in the nonpreemptive case. The subjective part of this procedure is the division of the goals into priority sets and the selection of penalties within a priority set.

Optimum Portfolio Problem

A mutual fund manager has $200 million to invest and is considering five alternative investments. A portfolio is defined by specifying the number of units of each opportunity purchased. Each investment has a fixed unit cost, but its annual return is a random variable. Therefore, its value is not known with certainty. The research department has determined that the expected return and variance per unit of investment is proportional to the number of units invested in the opportunity. All the data are shown in Table 1. Costs and annual returns are given in millions of dollars.

The total expected return is the sum of the expected returns of the individual investments. Similarly, assuming independence, the total variance is the sum of the individual variances. The following three goals have been established for the portfolio and are listed in priority order.

Goal 1: The annual expected return must be at least \$45 million.

Goal 2: The total variance must be no more than \$150 million^2.

Goal 3: The amount invested in opportunities 2 and 4 should be equal.

The \$200 million budget is a hard constraint. A preemptive goal programming approach is to be used.

Table 1. Unit data for investment opportunities

Investment #	1	2	3	4	5
Cost, \$	20	60	30	65	30
Expected return, \$	6	15	6	12	5
Variance, \$^{2}	40	50	20	30	20

Model

For the linear programming model we let x_j be the number of units of investment j purchased. The first technological constraint below limits the amount of money invested while the next three reflect the three goals. We have added constraints to compute the value of the return R and variance V.

$$
\begin{array}{rlcr}
\text{Budget:} & 20x_1 + 60x_2 + 30x_3 + 65x_4 + 30x_5 & \le & 200 \\
\text{G1:} & 6x_1 + 15x_2 + 6x_3 + 12x_4 + 5x_5 - y_1^+ + y_1^- & = & 45 \\
\text{G2:} & 40x_1 + 50x_2 + 20x_3 + 30x_4 + 20x_5 - y_2^+ + y_2^- & = & 150 \\
\text{G3:} & 60x_2 - 65x_4 - y_3^+ + y_3^- & = & 0 \\
\text{Return:} & 6x_1 + 15x_2 + 6x_3 + 12x_4 + 5x_5 - R & = & 0 \\
\text{Variance:} & 40x_1 + 50x_2 + 20x_3 + 30x_4 + 20x_5 - V & = & 0
\end{array}
$$

$$x_j \ge 0,\ \ j = 1,\ldots,5;\ \ y_k^+ \ge 0 \text{ and } y_k^- \ge 0,\ \ k = 1,2,3;\ \ R \ge 0,\ \ V \ge 0$$

G1 is a lower one-sided goal so we adopt the penalties $p_1^- = 1$ and $p_1^+ = 0$, and solve the linear programming problem

$$\text{Minimize } z_1 = y_1^- - 0.001R$$

subject to the above constraints. Although not really necessary, we have added the term involving R to the objective function so that the solution will deliver the largest return that satisfies G1. The solution obtained is x_1 =10, y_1^+ = 15 and y_2^+ = 210, with all other variables 0. Goal 1 is satisfied with a return of 60, larger than the goal of 45 (y_1^+ = 15). Goal 2 is not satisfied because the variance is 400, much larger than the goal of 150 (y_2^+ = 250). Goal 3 is satisfied because both x_1 and x_2 are zero.

At the second iteration we add a constraint to keep the first goal at the value obtained in the first iteration:

$$y_1^- = 0.$$

Since G2 is an upper one-sided goal, we use the penalties p_2^- = 0 and p_2^+ = 1, and solve the linear program

$$\text{Minimize } z_2 = y_2^+ - 0.001R + 0.001V$$

subject to the above constraints. We have added terms in R and V to encourage a large return and a small variance. This time the solution does not use x_1 but has x_2 =2.06 and x_4 = 1.18. The corresponding return exactly meets the first goal of 45. The values of the second set of deviation variables are y_2^- = 11.8 and y_2^+ = 0 which indicate that the second goal is exceeded with a variance of 138.2. For the third goal, the solution y_3^+ = 47.05 indicates that there is a difference in the investments in opportunities 2 and 4. All other variables are 0.

At the third iteration we add another constraint to keep the second goal satisfied at its current value; i.e.,

$$y_2^+ = 0.$$

G3 is a two-sided goal so we use equal penalties p_3^- = 1 and p_3^+ = 1, and solve the linear program

$$\text{Minimize } z_3 = y_3^+ + y_3^- - 0.001R + 0.001V$$

subject to the above constraints. Rounded to one decimal point, the solution now calls for investment in three of the opportunities: x_1 =0.7, x_2 = 1.7, x_4 = 1.3. All deviation variables are 0 except y_3^+ = 13.3. The goals for return and variance are exactly met (R = 45 and V = 150) while the goal associated with the amounts invested in opportunities 2 and 4 is within 13.3 of being reached. This is the best solution possible given the preemptive nature of the priorities. The results for iteration 3 are summarized in Table 2.

Table 2. Summary of results for final iteration

Opportunity #	1	2	3	4	5	Total
Units bought	0.7	1.7	0	1.3	0	3.7
Cost, $	13.3	100.0	0	86.7	0	200
Expected return, $	4.0	25.0	0	16.0	0	45
Variance, $\$^2$	26.7	83.3	0	40.0	0	150

A.4 Fractional Programming

A number of situations arise when it is desirable to optimize the ratio of two functions. In productivity analysis, for example, one wishes to maximize the ratio of worker output to labor-hours expended to perform a task. In financial planning it is common to maximize the ratio of the expected return of a portfolio to the standard deviation of some measure of performance. When the two functions are linear, and the decision variables are defined over a polyhedral set, we get the following fractional programming problem

$$\text{Maximize } \; f(\mathbf{x}) = \frac{c_0 + \mathbf{cx}}{d_0 + \mathbf{dx}}$$

$$\text{subject to } \; \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$$

where c_0 and d_0 are scalars, and **c** and **d** are n-dimensional row vectors of coefficients

Under certain conditions, this optimization problem can be transformed into a linear program. In particular, we will assume that x_j is so restricted that the denominator of the fraction is strictly positive and that the maximum of $f(\mathbf{x})$ is finite; that is, $d_0 + \mathbf{dx} > 0$ and $f(\mathbf{x}) < \infty$ for all $\mathbf{x}$ in $\{\mathbf{x} : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. To put the problem into a more manageable form, we define the variable t as

$$t \equiv \frac{1}{d_0 + \mathbf{dx}}$$

and write the objective function as

$$f(\mathbf{x}) = c_0 t + \mathbf{cx}t.$$

By assumption, $t > 0$ for all feasible x_j. We now make the following change of variables.

$$y_j = x_j t \text{ or in vector notation, } \mathbf{y} = \mathbf{x}t.$$

Thus the transformed model becomes the linear program

$$\begin{aligned} \text{Maximize } \quad & c_0 t + \mathbf{cy} \\ \text{subject to } \quad & d_0 t + \mathbf{dy} = 1 \\ & \mathbf{Ay} - \mathbf{b}t = \mathbf{0} \\ & \mathbf{y} \geq \mathbf{0},\ t \geq 0 \end{aligned}$$

Note that it is permissible to restrict t to be greater than or equal to zero because of our assumptions. More generally, when $f(\mathbf{x}) = f_1(\mathbf{x})/f_2(\mathbf{x})$the same kind of transformation can be used to convert a fractional program with concave $f_1(\mathbf{x})$ and convex $f_2(\mathbf{x})$ to an equivalent convex program.

A.5 The Complementarity Problem

When investigating the quadratic programming problem in the nonlinear programming methods chapter, we show that it can be written as a series of linear equations in nonnegative variables subject to a set of *complementarity constraints* of the form $x_j y_j = 0$ for all j. A specialized linear programming algorithm can then be used to find a solution. To describe the more general situation, suppose we are given the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. The *complementarity problem* is to find a *feasible* solution for the set of constraints

$$\mathbf{y} = F(\mathbf{x}),\ \ \mathbf{x} \geq \mathbf{0},\ \ \mathbf{y} \geq \mathbf{0}$$

that also satisfies the complementarity constraint

$$\mathbf{x}^{\mathrm{T}}\mathbf{y} = 0$$

where F is a given vector-valued function. The problem has no objective function so technically it is not a full-fledged nonlinear program. It is called the complementarity problem because of the requirements that either

$$x_j = 0 \text{ or } y_j = 0 \text{ (or both) for all } j = 1, 2, \ldots, n.$$

An important special case, which includes the quadratic programming problem, is the *linear complementarity problem* (LCP) where $F(\mathbf{x}) = \mathbf{q} + \mathbf{M}\mathbf{x}$. Here, $\mathbf{q}$ is a given column vector and $\mathbf{M}$ is a given $n \times n$ matrix. Efficient algorithms have been developed for solving the LCP under suitable assumptions about the properties of the matrix $\mathbf{M}$. The most common approach involves pivoting from one basic feasible solution to the next, much like the simplex method. Linear and nonlinear applications of the complementarity problem can be found in game theory, engineering, and the computation of economic equilibria.

A.6 Exercises

1. For the nonlinear objective function example in Section 3.1, change the breakpoints for the linear approximation as follows: breakpoints for x_1 are 0, 2 and 3 and breakpoints for x_2 are 0, 2 and 4. Set up and solve the resultant LP approximation.

2. The operations manager of an electronics firm wants to develop a production plan for the next six months. Projected orders for the company's products are listed below along with the direct cost of production in each month. The plan must specify the monthly amount to produce so that all demand is met. Shortages are not permitted. Any amount produced in excess of demand can be stored in inventory for later use at a cost of \$4/unit/mo. Initial and final inventories are 0.

Month	Demand (units)	Production cost (\$/unit)
1	1300	100
2	1400	105
3	1000	110
4	800	115
5	1700	110
6	1900	110

   In addition to the direct costs of production and inventory, overhead costs must be charged for the maximum production level obtained and the maximum inventory level obtained during the 6-month period. The following information should be used.

   (i) Overhead cost for production = \$300 × (maximum production level).

   (ii) Overhead cost for inventory = \$100 × (maximum level of inventory).

   These costs are charged only once during the 6-month period. Set up and solve the linear programming model that determines the minimum cost plan.

3. Consider the situation described in Exercise 2. Rather than being concerned about the overhead costs of production and inventory, it is decided that the problem will be solved with a goal programming approach. The following goals have been established.

   G1: The average production cost is to be no more than \$109 per unit.

   G2: The maximum monthly production level in the six months is to be 1500 units.

   G3: The maximum monthly inventory level in the six months is to be 100 units.

   a. Assuming that the priority of the goals is in the order given, use the preemptive sequential procedure to solve the problem.

b. Assuming the priority of the goals is the reverse of the order given, use the preemptive procedure to solve this problem.

4. For each of the objective functions listed below, explain whether or not a piecewise linear approximation solved with a linear programming code will yield an acceptable solution. In all cases, the variables are restricted to be nonnegative.

a. Maximize $f(\mathbf{x}) = -2x_1^2 + 4x_1 x_2 - 4x_2^2 + 4x_1 + 4x_2$

b. Maximize $f(\mathbf{x}) = \ln(x_1 + 1) + \ln(x_2 + 1)$

c. Maximize $f(\mathbf{x}) = x_1^2 + 4_2^2 - 10x_1 + 20x_2$

d. Minimize $f(\mathbf{x}) = x_1^2 + 4_2^2 - 10x_1 + 20x_2$

e. Minimize $f(\mathbf{x}) = \sum_{j=1}^{n} f_j(x_j)$, where $f_j(x_j) = a_j(x_j)^{-b}$, with $a_j > 0$ and $0 < b < 1$

f. Maximize $f(\mathbf{x}) = \sum_{j=1}^{n} r_j(x_j) - \sum_{j=1}^{n} c_j(x_j)$

where $r_j = a_j\left(1 - \exp(-b_j x_j)\right)$ with $a_j > 0$ and $b_j > 0$;

$c_j = d_j(x_j)^b$ with $d_j > 0$ and $b > 1$

5. Develop a with a piecewise linear approximation for the nonlinear objective function in the problem given below and solve with an LP code. Use the integers as breakpoints.

$$\text{Maximize } z = -(x_1 - 4)^2 - (x_2 - 4)^2 - (x_3 - 4)^2$$

$$\begin{aligned} \text{subject to} \quad & x_1 + x_2 + x_3 \geq 1 \\ & x_1 + x_2 + x_3 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

6. Consider a problem of the form

$$\text{Minimize } \sum_{j=1}^{n} c_j|x_j|$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}x_j \geq b_i, \quad i = 1,\ldots,m$$

where the decision variables x_j are unrestricted and the cost coefficients c_j are all nonnegative. The objective function comprises the sum of piecewise linear terms $c_j|x_j|$ and can be shown, with not much difficulty, to be convex.

a. Convert the above problem to a linear program by making use of the ideas in the section on "maximizing the minimum of several linear functions."

b. Alternatively, convert the above problem to a linear program by making use of the fact that any unrestricted variable can be replaced with the difference of two nonnegative variables. That is, for x_j unrestricted, we can make the substitution $x_j \equiv x_j^+ - x_j^-$, where $x_j^+ \geq 0$ and $x_j^- \geq 0$. Note that to achieve the desired result, more than a direct substitution is required.

7. You are given m data points of the form $(\mathbf{a}_i, b_i)$, $i = 1,\ldots,m$, where $\mathbf{a}_i$ is an n-dimensional row vector and b_i is a scalar, and wish to build a model to predict the value of the variable b from knowledge of a specific vector $\mathbf{a}$. In such a situation, it is common to use a linear model of the form $b = \mathbf{ax}$, where $\mathbf{x}$ is an n-dimensional parameter vector to be determined. Given a particular realization of the vector $\mathbf{x}$, the *residual*, or prediction error, associated with the ith data point is defined as $|b_i - \mathbf{a}_i\mathbf{x}|$. Your model should "explain" the available data as best as possible; i.e., produce small residuals.

   a. Develop a mathematical programming model that minimizes the maximum residual. Convert your model to a linear program.

   b. Alternatively, formulate a model that minimizes the sum of the residuals. Convert this model to a linear program.

8. A government agency has five projects that it wishes to outsource. After publishing an announcement containing a request for proposals to perform the work, it received bids from three contractors. The bids are shown in the table below. The goal of the agency is to minimize its total cost. Set up and solve the linear programming model under the following conditions.

   a. Each contractor can perform as many as two projects; all five projects must be done.

   b. Each contractor can perform only one project and as many projects as possible should be done.

   c. There is no limit to the number of projects a contractor can perform and all projects must be done.

Cost of completing the different projects ($1000)

	Project 1	Project 2	Project 3	Project 4	Project 5
Contractor 1	65	37	42	29	28
Contractor 2	59	39	50	29	31
Contractor 3	62	46	33	24	31

9. A production scheduling problem must be solved over a 12-month period. All quantities are to be determined on a monthly basis. Parameter definitions and related conditions are stated below. Indices on the parameters and variables range from 1 to 12.

   - Demand in month i is d_i. This demand must be satisfied in each month so shortages are not allowed.
   - Cost of production in month i is p_i. The maximum production in month i is M_i.
   - Items produced may be either shipped to meet demand or held in inventory to meet demand in a subsequent month.
   - The inventory level at the end of month 12 must equal the initial inventory level at time zero (when the time horizon begins). This quantity is to be determine in the solution process.
   - The maximum amount that can be stored from one month to the next is 16 units.

   Problem statement: Develop a model to determine the optimal production quantity x_i in each month ($i = 1,\dots,12$), and the optimal amount to store in inventory in each month y_i so that total cost is minimized. Write the model for the four cases below. Each should be answered independently of the others.

   a. The cost of inventory is proportional to the amount stored. The cost is h dollars per unit per month.

   b. The cost of inventory is a nonlinear function of the amount stored. The total inventory cost in month i is $h(y_i) = a(y_i)^2$, where a is a constant and has dimensions of dollars per month. Use a piecewise linear approximation with the breakpoints taken as the powers of 2 (i.e., 0, 2, 4, 8, 16).

   c. The cost of inventory depends on how many months a unit is stored, and grows exponentially. The cost for one month is a dollars per unit, the cost for two months is $4a$ per unit, the cost for three months is $8a$ per unit, and so on.

d. Ignore the cost of inventory and the limit on the maximum amount that can be stored, and minimize the maximum inventory over the planning period.

10. A company manufactures two products, X and Y, from a mix of chemicals. The products are sold by the pound. Up to 1000 pounds of X can be sold for $12/lb but the price must be reduced to $9/lb for sales in excess of 1000 lb up to a maximum of 3000 lb in total sales. Product Y is sold for $18/lb for any amount up to 2000 lb. If more than 3000 lb of X or more than 2000 lb of Y are produced, the excess must be discarded at a cost of $2/lb.

    After processing the mix, the products are withdrawn in the following proportions: 40% is X, 20% is Y, and 40% is waste that must be discarded at a cost of $2/lb. Processing costs are $1.50/lb.

    The mix is made up of three raw materials identified by the letters A, B and C, and must be at least 45% raw material A and no more than 30% C. Raw material C is free for up to 1500 lb. Material C costs $4.50/lb for amounts above 1500 lb. No more than 3000 lb of material C is available at any price. Material A costs $6/lb for any amount. There is no limit to the amount of A that can be purchased. Material B costs $3/lb up to 2500 lb and $5.50/lb for additional quantities up to a total of 4000 lb.

    Write out and solve the linear programming model that will determine the production and sales plan that maximizes profit. Define all variables, describe each constraint, and indicate the transformations used to linearize any nonlinear functions.

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