Ch. 4 Sensitivity Analysis, Duality and Interior Point Methods Additional Exercises

12. The following linear program was solved with the big-M method.

$$\begin{aligned} \text{Minimize } z = 3x_1 + 6x_2 + x_3 - x_4 \\ \text{subject to } x_1 + x_2 + x_3 + x_4 = 12 \\ 4x_1 + 5x_2 + x_3 - x_4 \le 15 \\ x_j \ge 0, \quad j = 1,\ldots,4 \end{aligned}$$

The optimal tableau is shown below. Write the dual linear program and find its solution from the tableau.

Row	Basic variables	Coefficients							RHS
		z	x_1	x_2	x_3	x_4	x_{s1}	x_{s2}	
0	z	1	4	7	2	0	M+1	0	12
1	x_4	0	1	1	1	1	1	0	12
2	x_{s2}	0	5	6	2	0	1	1	27

14. Prove Theorem 6 that establishes complementary slackness.

15. Using matrix notation, write out the steps of the revised dual simplex algorithm. Use the dual simplex algorithm and the revised primal simplex algorithm in Chapter 3 as guides.

16. Consider the following integer program.

$$\begin{aligned} \text{Maximize } z = x_1 + x_2 \\ \text{subject to } -11x_1 + 4x_2 - x_3 = 4 \\ 2x_1 + x_2 + x_4 = 3 \\ -6x_1 + 4x_2 + x_5 = 7 \\ x_j \ge 0 \text{ and integer for } j = 1,\ldots,5 \end{aligned}$$

One way to approach an integer program is to begin by relaxing the integer requirements and solve the resulting linear program. When this is done for the problem above the optimal basic vector $\mathbf{x}_B = (x_1, x_2, x_3)$. Using the simplex tableau method or the revised simplex method, determine the corresponding solution and find the basis inverse.

The cutting plane method for integer programming adds a constraint to the linear program called a "cut." For this cut to be *valid*, it should make the current continuous solution infeasible without removing any feasible integer solution. A valid cut for this problem is the constraint

$$3x_4 + x_5 \geq 2$$

Using a surplus variable x_6, this constraint can be written as

$$-3x_4 - x_5 + x_6 = -2$$

Add this constraint to the problem and use either the revised or tableau form of the dual simplex algorithm to obtain a feasible solution. The variable x_6 should be the basic variable for the new row.

17. You are given the following integer program.

$$\begin{aligned} \text{Maximize } z &= -x_1 - x_2 \\ \text{subject to } \quad & 4x_1 + 10x_2 \geq 12 \\ & 10x_1 + 4x_2 \geq 12 \\ & x_1 \geq 0,\ x_2 \geq 0 \text{ and integer} \end{aligned}$$

By relaxing the integrality requirement and solving the resulting linear program, the optimal solution has $\mathbf{x}_B = (x_1, x_2)$ with

$$\mathbf{B}^{-1} = \begin{pmatrix} -4/84 & 10/84 \\ 10/84 & -4/84 \end{pmatrix}$$

Starting from this solution, sequentially add the cuts:

$$4x_1 + 9x_2 \geq 12 \qquad \text{(C1)}$$

$$4x_1 + 8x_2 \geq 12 \qquad \text{(C2)}$$

$$9x_1 + 4x_2 \geq 12 \qquad \text{(C3)}$$

$$8x_1 + 4x_2 \geq 12 \qquad \text{(C4)}$$

After each cut is added, reoptimize with either the revised or tableau form of the dual simplex algorithm. The final solution should have integer values for all the variables.

24. *Concave Quadratic Programming Problem* (Bertsimas and Tsitsiklis 1997)

Consider the following nonlinear optimization problem.

$$\text{Maximize} \quad \mathbf{cx} + \tfrac{1}{2}\mathbf{x}^T\mathbf{Qx}$$
$$\text{subject to} \quad \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$$

where $\mathbf{Q}$ is an $n \times n$ negative semidefinite matrix (that is, $\mathbf{x}^T\mathbf{Qx} \leq 0$ for all $\mathbf{x}$). The associated logarithmic barrier problem is

$$\text{Maximize} \quad \mathbf{cx} + \tfrac{1}{2}\mathbf{x}^T\mathbf{Qx} + \mu\sum_{j=1}^{n}\log(x_j)$$
$$\text{subject to} \quad \mathbf{Ax} = \mathbf{b}$$

The corresponding optimality conditions for this problem are

$$\begin{aligned} \mathbf{Ax}(\mu) &= \mathbf{b} \\ -\mathbf{Qx}(\mu) + \mathbf{A}^T\pi(\mu) - \mathbf{z}(\mu) &= \mathbf{c}^T \\ \mathbf{X}(\mu)\mathbf{Z}(\mu)\mathbf{e} &= \mu\mathbf{e} \end{aligned}$$

where $\mathbf{X}(\mu) = \text{diag}\{x_1(\mu),\ldots,x_n(\mu)\}$, $\mathbf{Z}(\mu) = \text{diag}\{z_1(\mu),\ldots,z_n(\mu)\}$ and $\mathbf{e} = (1,\ldots,1)^T$.

a. Apply the ideas in Section 4.3 to show that the Newton equations for this nonlinear system with μ fixed are:

$$\begin{aligned} \mathbf{Ad}_x &= \mathbf{0} \\ -\mathbf{Qd}_x + \mathbf{A}^T\mathbf{d}_\pi - \mathbf{d}_z &= \mathbf{0} \\ \mathbf{Zd}_x + \mathbf{Xd}_z &= \mu\mathbf{e} - \mathbf{XZe} \end{aligned}$$

Assume that an initial interior point is available.

b. Work through the algebra to show that the solution to the linear system in part (a) is given by

$$\mathbf{d}_\pi = (\mathbf{A}(\mathbf{Z} - \mathbf{XQ})^{-1}\mathbf{XA}^T)^{-1}\mathbf{A}(\mathbf{Z} - \mathbf{XQ})^{-1}(\mu\mathbf{e} - \mathbf{XZe})$$

$$\mathbf{d}_z = -\mathbf{X}^{-1}(\mathbf{Zd}_x - \mu\mathbf{e} + \mathbf{XZe})$$

$$\mathbf{d}_x = (\mathbf{Z} - \mathbf{XQ})^{-1}(\mu\mathbf{e} - \mathbf{XZe} - \mathbf{XA}^T\mathbf{d}_\pi)$$

c. Based on the results in part (b), develop a primal-dual path following interior point algorithm to solve the original quadratic program.

25. Develop the equivalent of the Newton equations (7) and Newton directions (8) - (10) for the liner program given with inequality constraints; i.e.,

$$\begin{array}{ll} \text{Maximize} & \mathbf{cx} \\ \text{subject to} & \mathbf{Ax} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \end{array}$$

To begin, add slacks, write the dual and then formulate the necessary conditions as in Table 11.